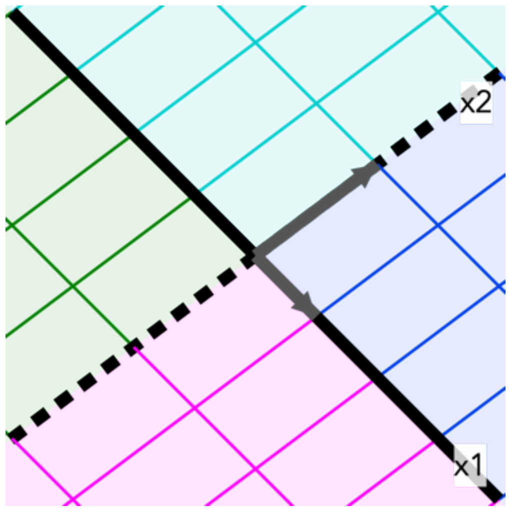
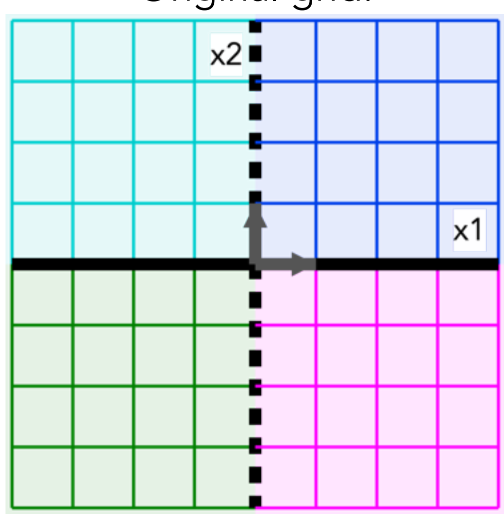
How does this work?

Original (identity) grid transformed by a full-rank matrix M =

$$\begin{bmatrix} 1 & 2 \\ -1 & 1.5 \end{bmatrix}$$



Original grid:



As a result of the transformation,

- Vector e1 = $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ moves to column 1 of M = $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

• Vector e2 = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ moves to column 2 of M = $\begin{bmatrix} 2 \\ 1.5 \end{bmatrix}$

• All other vectors are transformed
by combining these two axis directions

What do the steps show?

The 'Steps' table shows reconstruction of matrix M
from the identity matrix I (square grid)

This is only possible when M is full-rank

Recall:

• the reduced echelon form of any full-rank n×n matrix is I

• row reduction is equivalent to sequential left-multiplication
by elementary matrices $E_1 \dots E_k$
(k = number of row-reduction steps)

$$\Downarrow$$

$$(E_k \dots E_1)\, M = I$$

$$\Leftrightarrow$$

$$M = (E_k \dots E_1)^{-1}\, I = E_1^{-1} \dots E_k^{-1}\, I$$

Each image in the table represents
sequential left-multiplication of I (the grid)

by $E_1^{-1} \dots E_k^{-1}$

Scaling

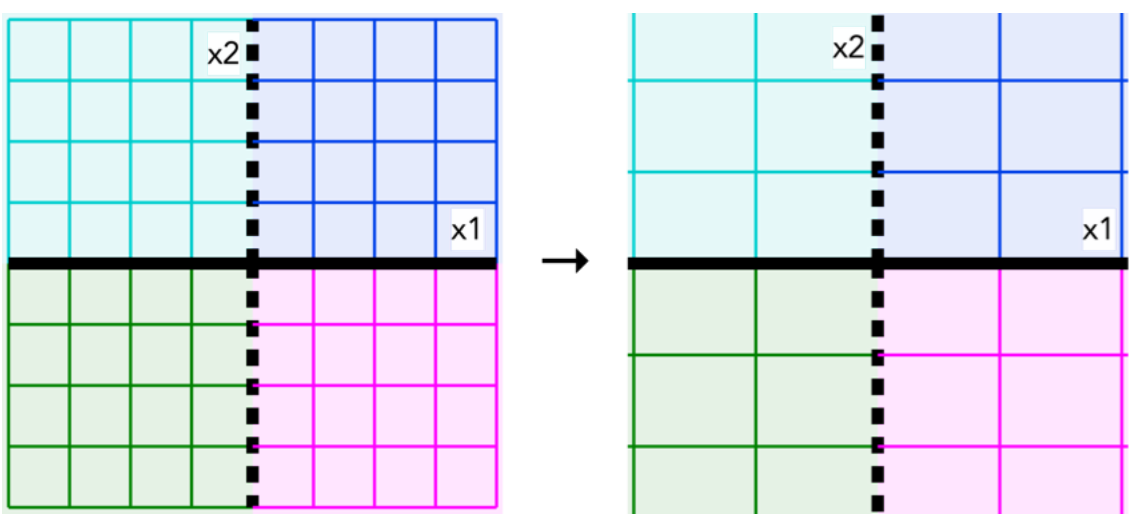
has the following form:

$$\begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

Matrix example:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}$$

$x_1' = s_1 x_1, \quad x_2' = s_2 x_2$



Shear in x_1 direction

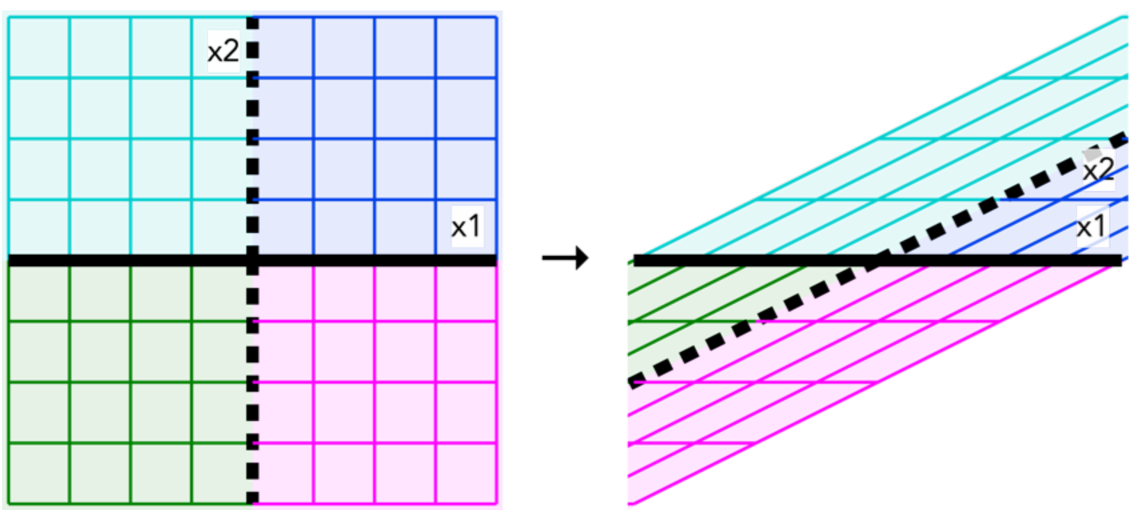
has the following form:

1	s
0	1

Matrix example:

1	2
0	1

$$x_1' = x_1 + sx_2, \quad x_2' = x_2$$



Shear in x_2 direction

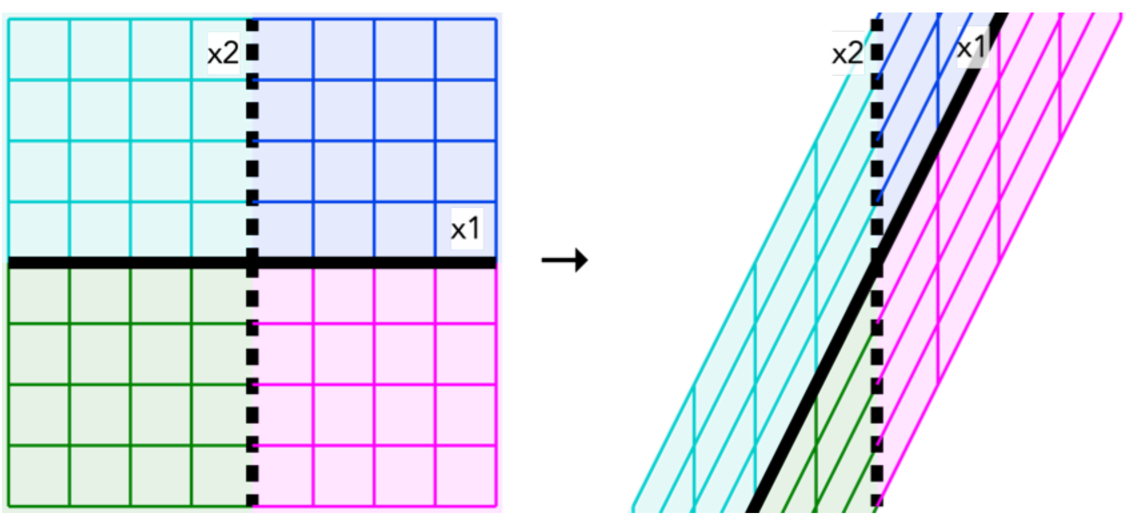
has the following form:

$$\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

Matrix example:

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$x_1' = x_1, \quad x_2' = sx_1 + x_2$$



Rotation

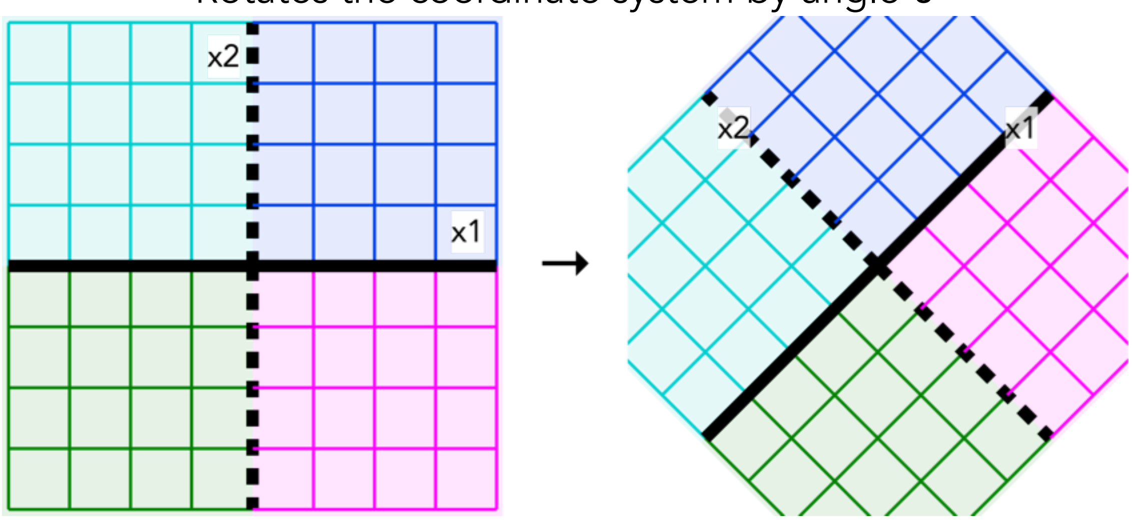
has the following form:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Matrix example $\left(\theta = \frac{\pi}{4}\right)$:

$$\begin{bmatrix} 0.70711 & -0.70711 \\ 0.70711 & 0.70711 \end{bmatrix}$$

Rotates the coordinate system by angle θ



- Columns of M are orthogonal

⇔

$M^T \times M$ is diagonal

- Each column of M has length 1

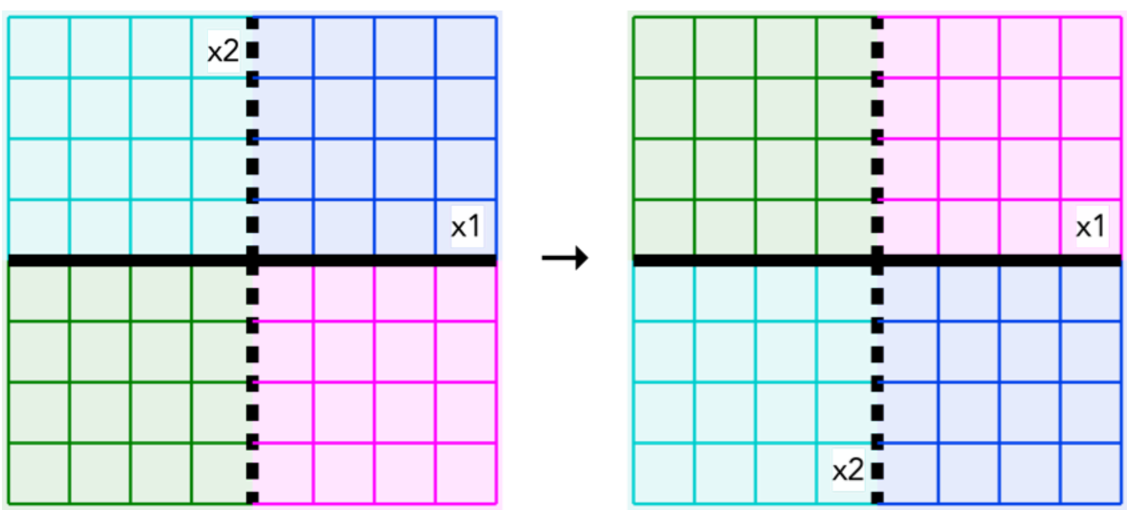
⇔

$M^T \times M = I$

Reflection across x_1 axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$(x_1, x_2) \rightarrow (x_1, -x_2)$



- Columns of M are orthogonal

⇔

$M^T \times M$ is diagonal

- Each column of M has length 1

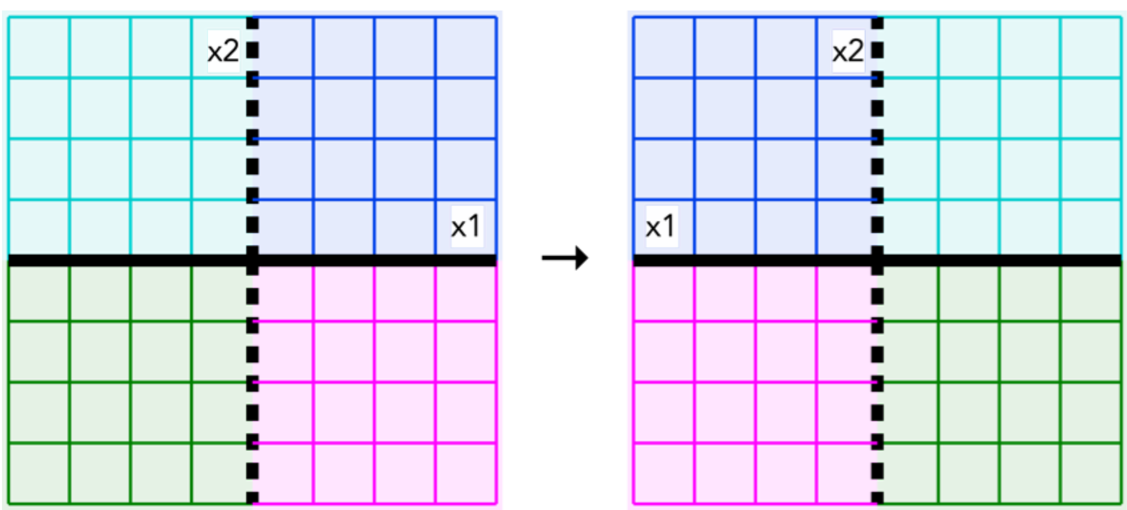
⇔

$M^T \times M = I$

Reflection across x_2 axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$(x_1, x_2) \to (-x_1, x_2)$



- Columns of M are orthogonal

⇔

$M^T \times M$ is diagonal

‣ Each column of M has length 1

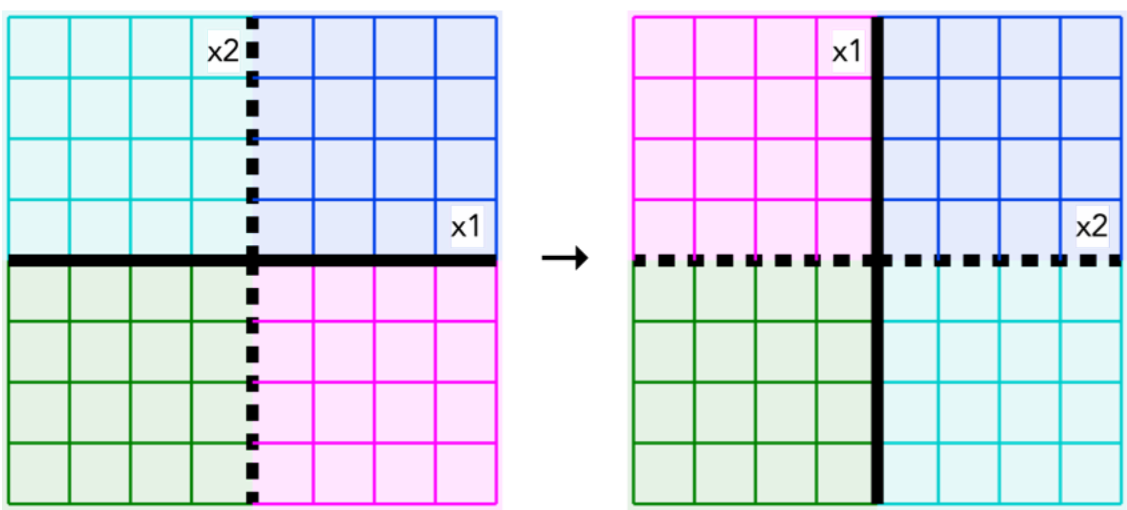
⇔

$M^T \times M = I$

Reflection across line $x_2 = x_1$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$(x_1, x_2) \rightarrow (x_2, x_1)$



‣ Columns of M are orthogonal

⇔

$M^T \times M$ is diagonal

‣ Each column of M has length 1

$\Leftrightarrow$

$M^{T} \times M = I$

Rotation + scaling

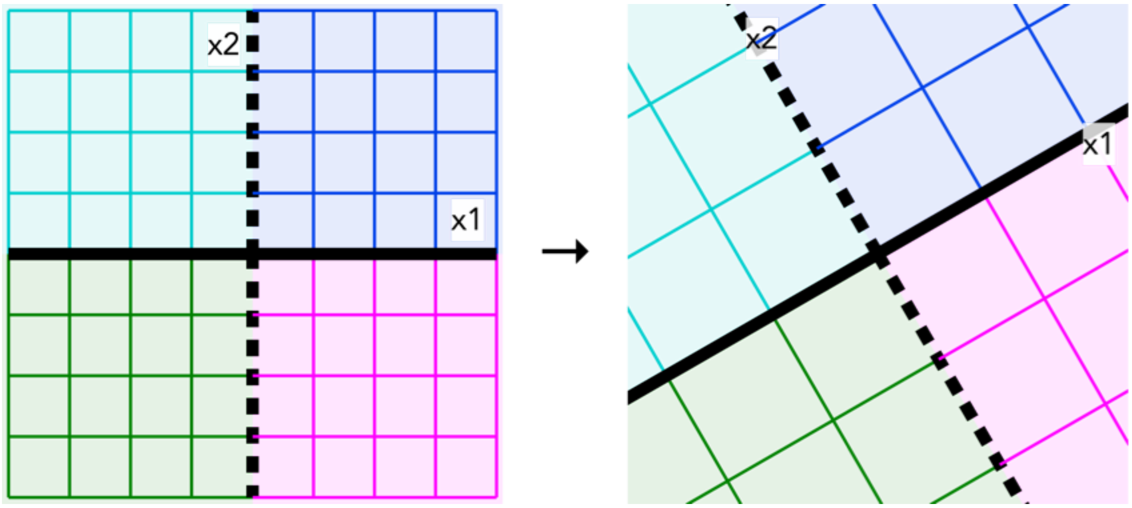
has the following form:

$$\begin{bmatrix} s\cos(\theta) & -s\sin(\theta) \\ s\sin(\theta) & s\cos(\theta) \end{bmatrix}$$

Matrix example $\left(\theta = \frac{\pi}{6}\right)$:

$$\begin{bmatrix} 1.73205 & -1 \\ 1 & 1.73205 \end{bmatrix}$$

- Rotates the coordinate system by angle θ
- Scales all directions by s



‣ Columns of M are orthogonal and same length

⇔

$$M^T \times M = s^2 I$$

Symmetric matrix

has the following form:

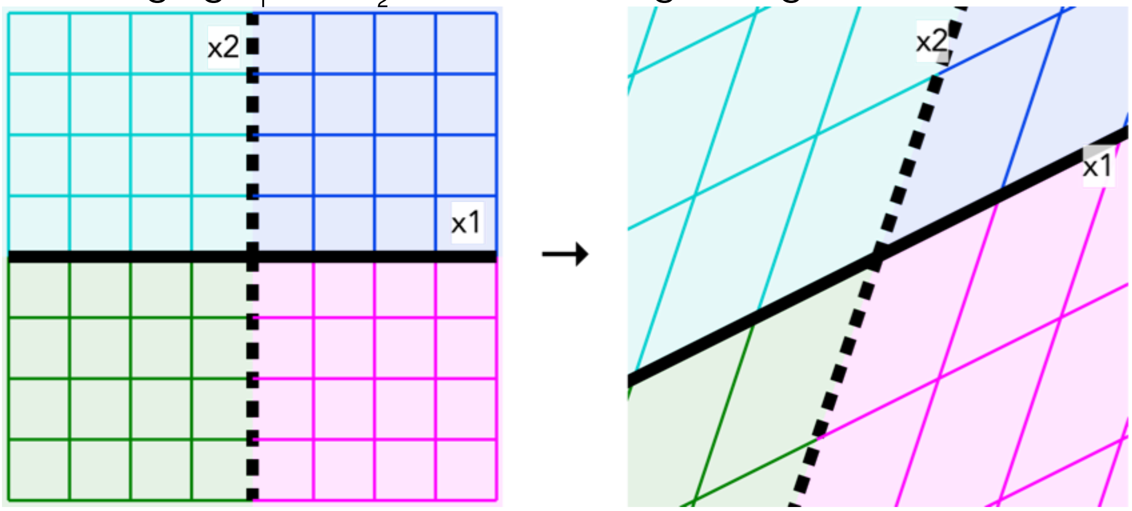
$$\begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

Matrix example:

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

• Off-diagonal effects are identical in both directions

• Exchanging x_1 and x_2 does not change the geometric distortion



Note the following general identity:

$(A \vec{x}) \cdot \vec{y} = \vec{x} \cdot (A^T \vec{y})$

If $A = A^T$, this becomes

$(A \vec{x})^T \vec{y} = \vec{x}^T A^T \vec{y} = \vec{x}^T (A^T \vec{y}) = \vec{x} \cdot (A^T \vec{y})$

Intuitively, this means the following:

• left side: apply A to $\vec{x}$, then measure its component in direction of $\vec{y}$

• right side: apply A to $\vec{y}$, then measure its component in direction of $\vec{x}$

• the two numbers are identical

Matrix transformation vs change of basis

Two ways to view a matrix:

① As transforming vectors, shown as grid transformation images

② As change in coordinate system where

basis vectors become columns of M

In the example shown previously, M =

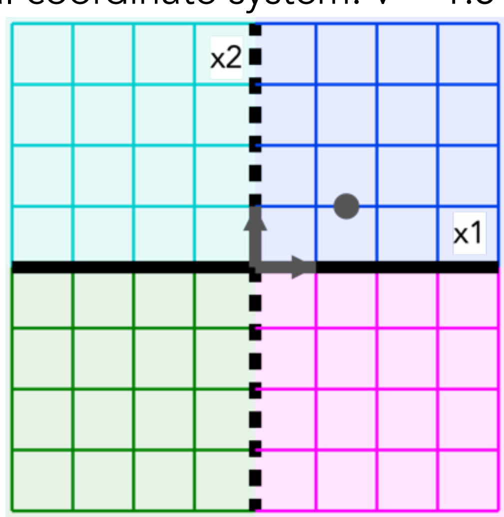
$$\begin{bmatrix} 1 & 2 \\ -1 & 1.5 \end{bmatrix}$$

- Basis vector $\vec{e}1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ becomes new basis vector $\vec{f}1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

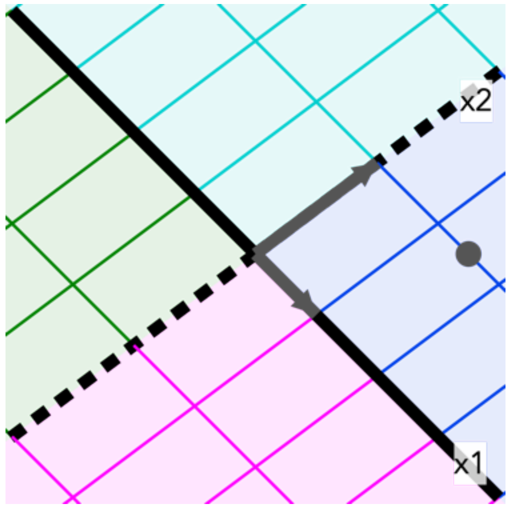
- Basis vector $\vec{e}2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ becomes new basis vector $\vec{f}2 = \begin{bmatrix} 2 \\ 1.5 \end{bmatrix}$

Consider example: vector or point $\vec{v} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$

$\vec{v}$ in original coordinate system: $\vec{v} = 1.5 \times \vec{e}1 + 1 \times \vec{e}2$



$\vec{v}$ in new system: $\vec{v} = 1.5 \times \vec{f}1 + 1 \times \vec{f}2$



The two images show the same vector $\vec{v}$
The difference between the images is the basis used to describe it

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Matrix-vector multiplication vs linear transformation

① Matrix–vector multiplication $\vec{u} = M\vec{v}$ is an algebraic operation defined so that $\vec{u}$ is a linear combination of the columns of M with coefficients from $\vec{v}$
(Thus, $M\vec{v}$ is valid when the number of columns of M equals the number of entries of $\vec{v}$)

Consider a 2×2 example:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} =$$

$$v_1 \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} + v_2 \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

② Linear transformation T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is a function that satisfies

two conditions for any $\vec{a}$ & $\vec{b} \in \mathbb{R}^n$:

- $T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b})$
- $T(s\vec{a}) = sT(\vec{a})$

This translates into the following geometric appearance

- Origin stays intact: $T(\vec{0}) = \vec{0}$
- Lines stay parallel and evenly spaced

③ Every matrix–vector multiplication is a linear transformation

Let M be an m×n matrix and define $T(\vec{a}) = M\vec{a}$

Then for any $\vec{a}$, $\vec{b} \in \mathbb{R}^n$ and any scalar s,

$$T(\vec{a} + \vec{b}) = M(\vec{a} + \vec{b}) = M\vec{a} + M\vec{b} = T(\vec{a}) + T(\vec{b})$$

$$T(s\vec{a}) = M(s\vec{a}) = s(M\vec{a}) = sT(\vec{a})$$

④ Every linear transformation becomes $M\vec{a}$ once a basis is chosen:

Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be linear, and choose the standard basis vectors

$\vec{e}_1, \ldots, \vec{e}_n$ in $\mathbb{R}^n$

Form the matrix M by using the images of the basis vectors as columns:

$$M = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

Now any vector $\vec{a} \in \mathbb{R}^n$ can be written as

$$\vec{a} = a_1\vec{e}_1 + \ldots + a_n\vec{e}_n = \sum_{j=1}^{n} \left(a_j\, \vec{e}_j\right)$$

$$T(\vec{a}) = T\left(\sum_{j=1}^{n} \left(a_j\, \vec{e}_j\right)\right)$$

Since T is linear, the transformation operator can be moved

inside the summation operator: $T\left(\sum_{j=1}^{n}\left(a_j\,\vec{e}_j\right)\right) = \sum_{j=1}^{n}\left(T\left(a_j\vec{e}_j\right)\right)$

and $T\left(a_j\,\vec{e}_j\right) = a_j\,T\left(\vec{e}_j\right)$

⇓

$$T(\vec{a}) = \sum_{j=1}^{n}\left(a_j\,T\left(\vec{e}_j\right)\right)$$

Since $M = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}$,

this sum is identical to how matrix–vector product $M\vec{a}$ is defined

⇓

- $T(\vec{a}) = M\vec{a}$ for all $\vec{a} \in \mathbb{R}^n$
- Columns of M are the directions that T forces basis vectors to

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One-to-one and onto linear transformations

① One-to-one (injective):

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called one-to-one if

$T(\vec{c}_1) = T(\vec{c}_2)$ implies $\vec{c}_1 = \vec{c}_2$

Suppose M is the m×n matrix representing T, so that
$T(\vec{c}) = M\vec{c}$

Then T is one-to-one if and only if
$M\vec{c} = \vec{0}$ implies $\vec{c} = \vec{0}$

Indeed, if $M\vec{c}_1 = M\vec{c}_2$, then
$M(\vec{c}_1 - \vec{c}_2) = \vec{0}$,
which implies $\vec{c}_1 - \vec{c}_2 = \vec{0}$,
and therefore $\vec{c}_1 = \vec{c}_2$

From the definition of matrix-vector multiplication,
$M\vec{c}$ is a linear combination of columns of M with coefficients from $\vec{c}$
Interpreting the one-to-one condition
$M\vec{c} = \vec{0}$ implies $\vec{c} = \vec{0}$
in terms of columns means:
we cannot obtain $\vec{0}$ as a nontrivial linear combination of columns of M
$\Leftrightarrow$
M has n linearly independent columns, or rank(M) = n

Geometric interpretation:
No dimension collapses and distinct points remain distinct

② Onto (surjective):

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called onto if

for every vector $\vec{u} \in \mathbb{R}^m$ there exists a vector $\vec{c} \in \mathbb{R}^n$ such that
$T(\vec{c}) = \vec{u}$

Suppose M is the m×n matrix representing T, so that
$T(\vec{c}) = M\vec{c}$

Then T is onto if and only if
for every vector $\vec{u} \in \mathbb{R}^m$, the equation $M\vec{c} = \vec{u}$ has a solution

From the definition of matrix-vector multiplication,
$M\vec{c}$ is a linear combination of columns of M with coefficients from $\vec{c}$
Interpreting the onto condition
for every $\vec{u} \in \mathbb{R}^m$, $M\vec{c} = \vec{u}$ has a solution
in terms of columns means:
every $\vec{u} \in \mathbb{R}^m$ can be written as a linear combination of columns of M
$\Leftrightarrow$
columns of M span $\mathbb{R}^m$, or rank(M) = m

Geometric interpretation:
The image of T fills the entire output space

Summary table:

Property	One-to-one (injective)	Onto (surjective)	Invertible
Definition	$T(c_1) = T(c_2) \Rightarrow c_1 = c_2$	For every $u \in \mathbb{R}^m$, Mc = u has a solution	One-to-one and onto
Matrix condition	$Mc = 0 \Rightarrow c = 0$	Columns of M span $\mathbb{R}^m$	Mc = u has a unique solution
Rank condition	rank(M) = n	rank(M) = m	rank(M) = n = m
Geometric meaning	No dimension collapses	Image fills output space	Perfect dimension match