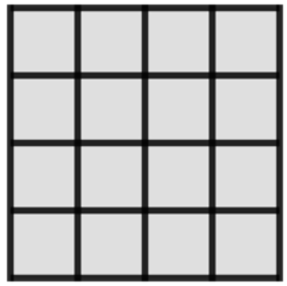
Square full-rank: $m = n$, $r = n$

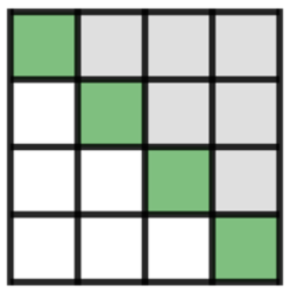
Consider matrix M of size 4×4 with rank $r = 4$
All columns are linearly independent



⦿ Domain(M): all vectors in $\mathbb{R}^4$

⦿ Codomain(M): all vectors in $\mathbb{R}^4$

⦿ Echelon form of M has the following appearance:



‣ Pivots are shown in green
‣ Forced zeros are shown in white
‣ There are $m-r = 0$ all-0 rows

⦿ Solution set for $M \vec{x} = \vec{0}$
same as null space (M)
same as input directions that get collapsed to $\vec{0}$
dimension = $n - r = 0$
⇔
$M \vec{x} = \vec{0}$ has only trivial solution
⇔
no free variables

⇔

if M $\vec{x}1$ = M $\vec{x}2$ then $\vec{x}1 = \vec{x}2$

⇔

transformation by M is one-to-one

⦿ row space (M)

set of all vectors orthogonal to null space:

dimension = r = 4

⦿ Set of all possible outputs of M $\vec{x}$

same as column space (M)

dimension = r = 4

⦿ Set of all vectors within codomain that are orthogonal to column space

same as ℓ-null space:

ℓ-null space = $\{\vec{0}\}$

⇩

column space = codomain

⇩

transformation by M fills codomain

⇔

transformation by M is onto

⦿ Solution set for M $\vec{x} = \vec{b}$

every $\vec{b}$ in codomain has at least one solution

⇔

when a solution exists, it is unique

(null space dimension n−r = 0)

Example shows RREF form of M with $\vec{b} \in$ column space(M)

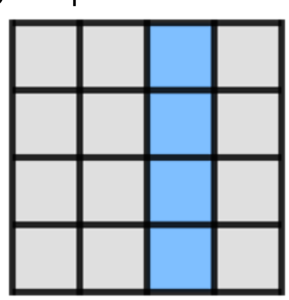
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & 0 & b_3 \\ 0 & 0 & 0 & 1 & b_4 \end{array}\right]$$

⦿ Determinant:
| M | is defined and non-zero

⦿ M^{-1} is defined

Square non-full-rank: $m = n$, $r < n$

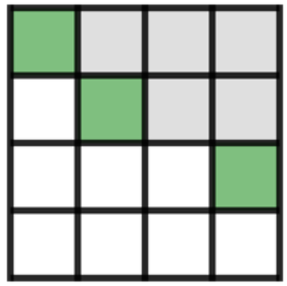
Consider matrix M of size 4×4 with rank $r = 3$
Column 3 is linearly dependent on preceding columns



⦿ Domain(M): all vectors in $\mathbb{R}^4$

⦿ Codomain(M): all vectors in $\mathbb{R}^4$

⦿ Echelon form of M has the following appearance:



‣ Pivots are shown in green

‣ Forced zeros are shown in white

‣ There are m-r = 1 all-0 rows

⦿ Solution set for $M \vec{x} = \vec{0}$

same as null space (M)

same as input directions that get collapsed to $\vec{0}$

dimension = n−r = 1

⇔

$M \vec{x} = \vec{0}$ has nontrivial solutions

⇔

free variables exist

⇔

there exist $\vec{x}1 \neq \vec{x}2$ such that $M \vec{x}1 = M \vec{x}2$

⇔

transformation by M is not one-to-one

⦿ row space (M)

set of all vectors orthogonal to null space:

dimension = r = 3

⦿ Set of all possible outputs of $M \vec{x}$

same as column space (M)

dimension = r = 3

⦿ Set of all vectors within codomain that are orthogonal to column space same as ℓ-null space:

dimension = m−r = 1

⇩

any $\vec{b}$ with nonzero ℓ-null component cannot be produced by M

⇔

transformation by M is not onto

⦿ Solution set for M $\vec{x} = \vec{b}$

- if $\vec{b} \notin$ column space(M): no solution
- if $\vec{b} \in$ column space(M): solutions exist

⇔

when a solution exists, solutions are not unique

free variable count n−r = 1

(same as dimension of null space)

unreachable dimension m−r = 1

Example shows RREF form of M with $\vec{b} \in$ column space(M)

$$\left[\begin{array}{cccc|c} 1 & 0 & m_{13} & 0 & b_1 \\ 0 & 1 & m_{23} & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Next example shows RREF form of M with $\vec{b} \notin$ column space(M)

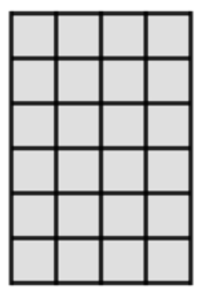
$$\left[\begin{array}{cccc|c} 1 & 0 & m_{13} & 0 & b_1 \\ 0 & 1 & m_{23} & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 0 & b_4 \neq 0 \end{array}\right]$$

⦿ Determinant:
$| M | = 0$

⦿ M^{-1} is undefined

Tall full column-rank: $m > n$, $r = n$

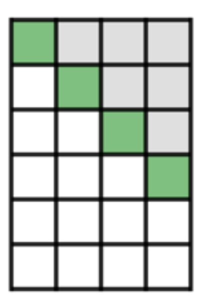
Consider matrix M of size 6×4 with rank $r = 4$
All columns are linearly independent



⦿ Domain(M): all vectors in $\mathbb{R}^4$

⦿ Codomain(M): all vectors in $\mathbb{R}^6$

⦿ Echelon form of M has the following appearance:



‣ Pivots are shown in green
‣ Forced zeros are shown in white
‣ There are m-r = 2 all-0 rows

⦿ Solution set for M $\vec{x} = \vec{0}$
same as null space (M)
same as input directions that get collapsed to $\vec{0}$
dimension = n−r = 0
⇔
M $\vec{x} = \vec{0}$ has only trivial solution
⇔
no free variables
⇔
if M $\vec{x}$1 = M $\vec{x}$2 then $\vec{x}$1 = $\vec{x}$2
⇔
transformation by M is one-to-one

⦿ row space (M)
set of all vectors orthogonal to null space:
dimension = r = 4

⦿ Set of all possible outputs of M $\vec{x}$
same as column space (M)
dimension = r = 4

⦿ Set of all vectors within codomain that are orthogonal to column space
same as ℓ-null space:
dimension = m−r = 2
⇓
any $\vec{b}$ with nonzero ℓ-null component cannot be produced by M
⇔
transformation by M is not onto

⦿ Solution set for M $\vec{x} = \vec{b}$

- if $\vec{b} \notin$ column space(M): no solution
- if $\vec{b} \in$ column space(M): solutions exist

$\Leftrightarrow$

when a solution exists, it is unique

(null space dimension $n-r = 0$)

unreachable dimension $m-r = 2$

Example shows RREF form of M with $\vec{b} \in$ column space(M)

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & 0 & b_3 \\ 0 & 0 & 0 & 1 & b_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Next example shows RREF form of M with $\vec{b} \notin$ column space(M)

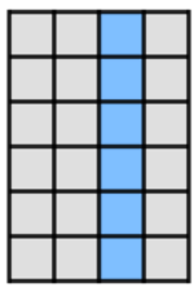
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & 0 & b_3 \\ 0 & 0 & 0 & 1 & b_4 \\ 0 & 0 & 0 & 0 & b_5 \neq 0 \\ 0 & 0 & 0 & 0 & b_6 \neq 0 \end{array}\right]$$

⦿ Determinant:
| M | is undefined

⦿ M^{-1} is undefined

Tall non-full column-rank: $m > n$, $r < n$

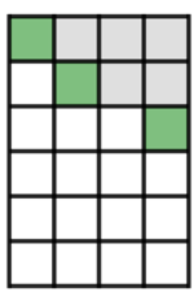
Consider matrix M of size 6×4 with rank $r = 3$
Column 3 is linearly dependent on preceding columns



⦿ Domain(M): all vectors in $\mathbb{R}^4$

⦿ Codomain(M): all vectors in $\mathbb{R}^6$

⦿ Echelon form of M has the following appearance:



‣ Pivots are shown in green
‣ Forced zeros are shown in white
‣ There are m-r = 3 all-0 rows

⦿ Solution set for M $\vec{x} = \vec{0}$
same as null space (M)
same as input directions that get collapsed to $\vec{0}$
dimension = n−r = 1
⇔
M $\vec{x} = \vec{0}$ has nontrivial solutions
⇔
free variables exist
⇔
there exist $\vec{x}1 \neq \vec{x}2$ such that M $\vec{x}1$ = M $\vec{x}2$
⇔
transformation by M is not one-to-one

⦿ row space (M)
set of all vectors orthogonal to null space:
dimension = r = 3

⦿ Set of all possible outputs of M $\vec{x}$
same as column space (M)
dimension = r = 3

⦿ Set of all vectors within codomain that are orthogonal to column space
same as ℓ-null space:
dimension = m−r = 3
⇓
any $\vec{b}$ with nonzero ℓ-null component cannot be produced by M
⇔
transformation by M is not onto

⦿ Solution set for M $\vec{x} = \vec{b}$

• if $\vec{b} \notin$ column space(M): no solution
• if $\vec{b} \in$ column space(M): solutions exist

⇔

when a solution exists, solutions are not unique

free variable count n−r = 1

(same as dimension of null space)

unreachable dimension m−r = 3

Example shows RREF form of M with $\vec{b} \in$ column space(M)

$$\left[\begin{array}{cccc|c} 1 & 0 & m_{13} & 0 & b_1 \\ 0 & 1 & m_{23} & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Next example shows RREF form of M with $\vec{b} \notin$ column space(M)

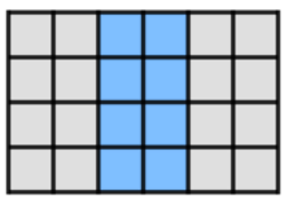
$$\left[\begin{array}{cccc|c} 1 & 0 & m_{13} & 0 & b_1 \\ 0 & 1 & m_{23} & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 0 & b_4 \neq 0 \\ 0 & 0 & 0 & 0 & b_5 \neq 0 \\ 0 & 0 & 0 & 0 & b_6 \neq 0 \end{array}\right]$$

⦿ Determinant:
| M | is undefined

⦿ M^{-1} is undefined

Short full row-rank: m < n, r = m

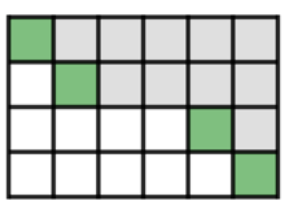
Consider matrix M of size 4×6 with rank r = 4
Columns 3, 4 are linearly dependent on preceding columns



⦿ Domain(M): all vectors in $\mathbb{R}^6$

⦿ Codomain(M): all vectors in $\mathbb{R}^4$

⦿ Echelon form of M has the following appearance:



‣ Pivots are shown in green
‣ Forced zeros are shown in white
‣ There are m-r = 0 all-0 rows

⦿ Solution set for M $\vec{x} = \vec{0}$
same as null space (M)
same as input directions that get collapsed to $\vec{0}$
dimension = n−r = 2
⇔
M $\vec{x} = \vec{0}$ has nontrivial solutions
⇔
free variables exist
⇔
there exist $\vec{x}1 \neq \vec{x}2$ such that M $\vec{x}1$ = M $\vec{x}2$
⇔
transformation by M is not one-to-one

⦿ row space (M)
set of all vectors orthogonal to null space:
dimension = r = 4

⦿ Set of all possible outputs of M $\vec{x}$
same as column space (M)
dimension = r = 4

⦿ Set of all vectors within codomain that are orthogonal to column space
same as ℓ-null space:
ℓ-null space = $\{\vec{0}\}$
⇩
column space = codomain
⇩
transformation by M fills codomain
⇔
transformation by M is onto

⦿ Solution set for $M \vec{x} = \vec{b}$

every $\vec{b}$ in codomain has at least one solution

⇔

when a solution exists, solutions are not unique

free variable count $n-r = 2$

(same as dimension of null space)

Example shows RREF form of M with $\vec{b} \in$ column space(M)

$$\left[\begin{array}{cccccc|c} 1 & 0 & m_{13} & m_{14} & 0 & 0 & b_1 \\ 0 & 1 & m_{23} & m_{24} & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & b_4 \end{array}\right]$$

⦿ Determinant:

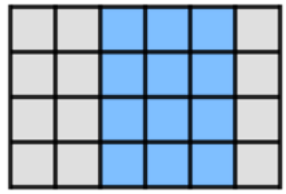
| M | is undefined

⦿ M^{-1} is undefined

Short non-full row-rank: $m < n$, $r < m$

Consider matrix M of size 4×6 with rank $r = 3$

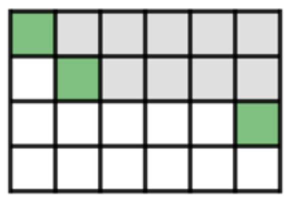
Columns 3, 4, 5 are linearly dependent on preceding columns



⦿ Domain(M): all vectors in $\mathbb{R}^6$

⦿ Codomain(M): all vectors in $\mathbb{R}^4$

⦿ Echelon form of M has the following appearance:



‣ Pivots are shown in green
‣ Forced zeros are shown in white
‣ There are m-r = 1 all-0 rows

⦿ Solution set for M $\vec{x} = \vec{0}$
same as null space (M)
same as input directions that get collapsed to $\vec{0}$
dimension = n−r = 3
⇔
M $\vec{x} = \vec{0}$ has nontrivial solutions
⇔
free variables exist
⇔
there exist $\vec{x}1 \neq \vec{x}2$ such that M $\vec{x}1$ = M $\vec{x}2$
⇔
transformation by M is not one-to-one

⦿ row space (M)
set of all vectors orthogonal to null space:
dimension = r = 3

⦿ Set of all possible outputs of M $\vec{x}$
same as column space (M)
dimension = r = 3

⦿ Set of all vectors within codomain that are orthogonal to column space
same as ℓ-null space:
dimension = m−r = 1
⇓
any $\vec{b}$ with nonzero ℓ-null component cannot be produced by M
⇔
transformation by M is not onto

⦿ Solution set for M $\vec{x}$ = $\vec{b}$
• if $\vec{b} \notin$ column space(M): no solution
• if $\vec{b} \in$ column space(M): solutions exist
⇔
when a solution exists, solutions are not unique
free variable count n−r = 3
(same as dimension of null space)
unreachable dimension m−r = 1

Example shows RREF form of M with $\vec{b} \in$ column space(M)

1	0	m_{13}	m_{14}	m_{15}	0	b_1
0	1	m_{23}	m_{24}	m_{25}	0	b_2
0	0	0	0	0	1	b_3
0	0	0	0	0	0	0

Next example shows RREF form of M with $\vec{b} \notin$ column space(M)

$$\left[\begin{array}{cccccc|c} 1 & 0 & m_{13} & m_{14} & m_{15} & 0 & b_1 \\ 0 & 1 & m_{23} & m_{24} & m_{25} & 0 & b_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_4 \neq 0 \end{array}\right]$$

⦿ Determinant:
| M | is undefined

⦿ M^{-1} is undefined