Block partition of square matrix

Many matrices contain structured blocks
for example, zero blocks, diagonal blocks or triangular blocks

When such structure is present, properties such as inverse or determinant can often be computed more directly by

- dividing the matrix into appropriate blocks
- computing the relevant properties of the individual blocks
- recombining the results to obtain the property for the full matrix

The Schur complement formalizes this idea for inverse and determinant computation

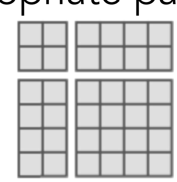
In this chapter, block partition is used for the purpose of determinant and inverse computation

Square matrix M is divided into 4 blocks:

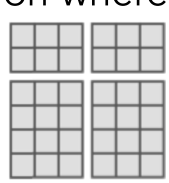
$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

For the purpose of this chapter, partitioning is appropriate
if diagonal blocks A and D are square,
which is necessary for the determinant and inverse of those blocks to be defined

One example of appropriate partition is shown below:



Compare this with invalid partition where diagonal blocks are non-square



Even though partition is done into 4 blocks,
it can be applied in a recursive fashion to the diagonal blocks

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Definition of Schur complement

Suppose M is a square matrix divided into blocks

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

in such a way that A is invertible

Schur complement of A in M is a matrix computed as
$S = D - CA^{-1}B$
(note that product $CA^{-1}B$ has same size as D, even if A and D have different sizes)

Suppose we perform a block row replacement operation on M to force the block C to 0
This would be accomplished by an elementary block matrix equivalent

$$L = \left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]$$

$$\text{Product } LM = \left[\begin{array}{c|c} A & B \\ \hline 0 & D - CA^{-1}B \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline 0 & S \end{array}\right]$$

Therefore, Schur complement S is the block that remains after eliminating C using A

Schur complement for determinant computation

Consider square matrix M divided as shown:

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where A is invertible

Recall the following:

1. Elementary block matrix $L = \left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]$

L is lower triangular

⇩

$|L| = |I| \times |I| = 1$

2. Product matrix $LM = \left[\begin{array}{c|c} A & B \\ \hline 0 & D - CA^{-1}B \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline 0 & S \end{array}\right]$

L is lower triangular

⇩

$$|LM| = |A| \times |S|$$

Determinant of product equals product of determinants

$$\Downarrow$$

$$|LM| = |L| \times |M|$$

$$\Downarrow$$

$$|M| = |A| \times |S|$$

Schur complement for M^{-1} computation

Consider invertible square matrix M divided as shown:

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where A is invertible

Will perform block row replacement operations on M, this time, to obtain RREF

Step 1: left-multiplication by block lower triangular matrix L
to force lower left block to 0

$$L = \left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]$$

$$\text{Product } LM = \left[\begin{array}{c|c} A & B \\ \hline 0 & D - CA^{-1}B \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline 0 & S \end{array}\right]$$

Step 2: left-multiplication by block diagonal matrix D
to normalize both diagonal blocks to I

$$D = \left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & S^{-1} \end{array}\right]$$

$$\text{Product } DLM = \left[\begin{array}{c|c} I & A^{-1}B \\ \hline 0 & I \end{array}\right]$$

Step 3: left-multiplication by block upper triangular matrix U
to force right upper block to 0

$$U = \left[\begin{array}{c|c} I & -A^{-1}B \\ \hline 0 & I \end{array}\right]$$

Product UDLM = I
Multiplying both sides by M^{-1}
gives us
$M^{-1} = UDL =$

$$\left[\begin{array}{c|c} I & -A^{-1}B \\ \hline 0 & I \end{array}\right]\left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & S^{-1} \end{array}\right]\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]$$

This procedure is ordinary row reduction applied at the block level,
with the Schur complement playing the role of the second pivot

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Summary

① We perform one row reduction step on M to force block (2, 1) to 0
Result is an upper triangular matrix LM with $S = D - CA^{-1}B$ as block (2, 2)

② For determinant computation, this is all we need
Determinant of the upper triangular LM is the product of determinants of its diagonal blocks
$|M| = |A| \times |S|$

③ For inverse computation, we continue the block row reduction all the way to RREF
We record each block elementary matrix used along the way
Reversing those steps reconstructs the inverse, just as in the standard elimination based inverse algorithm

Steps 1–3 above are the same steps used in the standard determinant and inverse algorithms
with the only exception of the “zero pivot” case
In ordinary row reduction we would swap rows to bring a nonzero pivot into place
At the block level, a swap may be impossible when block sizes do not match

So if A is not invertible, we start from the other diagonal block instead:

eliminate block (1,2) first using D (instead of eliminating block (2,1) using A)

Conclusion:

Schur complement is row reduction with constrained pivots rather than row swaps