Row view of a matrix

Previously, we showed that

- a matrix can be viewed as a collection of column vectors
- all linear combinations of those columns form all possible outputs

When discussing linear equations, we drew lines satisfying

$$\vec{a}_i \cdot \vec{x} = b_i$$

(for $b_i \neq 0$, these lines were shifted away from the origin)

Now we consider a non-augmented matrix A

Each row $\vec{a}_i$ defines a set

$$S_i = \{ \vec{x} \in \mathbb{R}^2 \mid \vec{a}_i \cdot \vec{x} = 0 \}$$

- Each S_i is a line through the origin
- Each S_i is the set of inputs sent to zero by that row

In the full-rank 2×2 case, S_1 and S_2 are different lines

Their intersection is {0}

In the non-full-rank case, the rows are dependent and $S_1 = S_2$

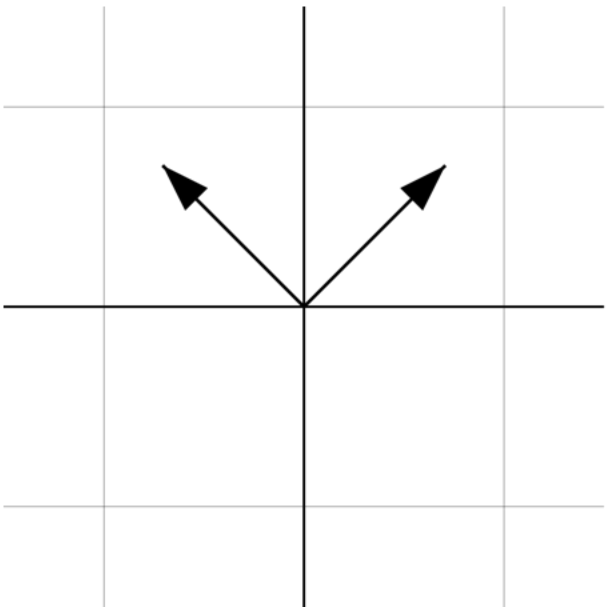
Their intersection is that entire line

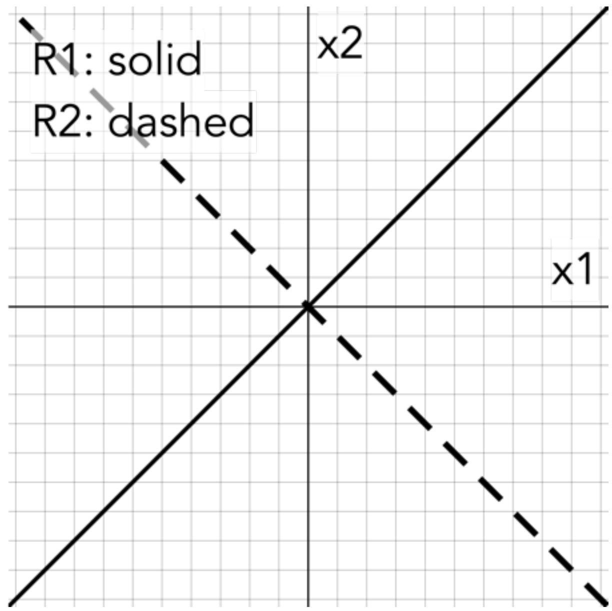
Images of row vs column views

- left image: columns as vectors
- right image: lines orthogonal to row vectors

① Rotation matrix (45°) = $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

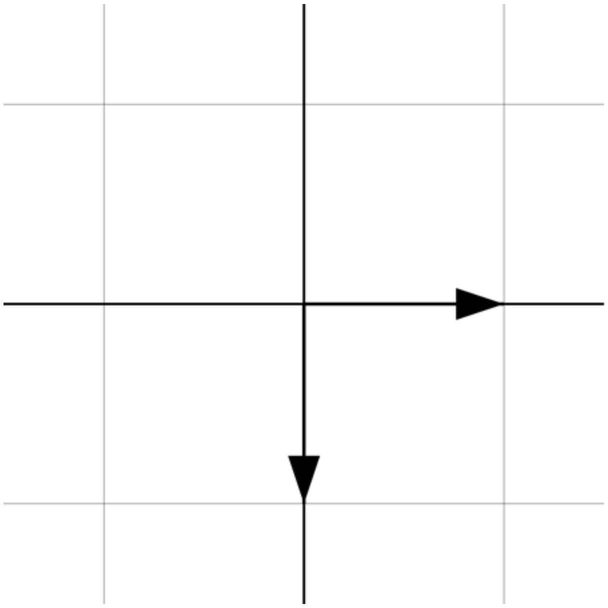
Orthonormal: constraints are perpendicular; intersection is {0}

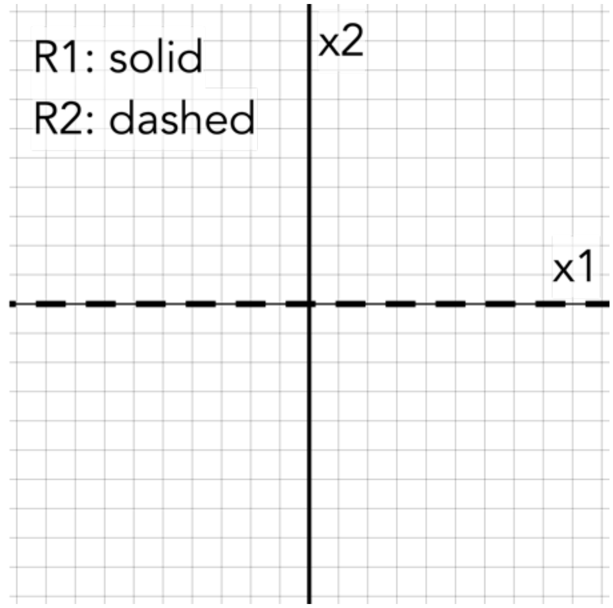




② Reflection across x_1-axis = $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

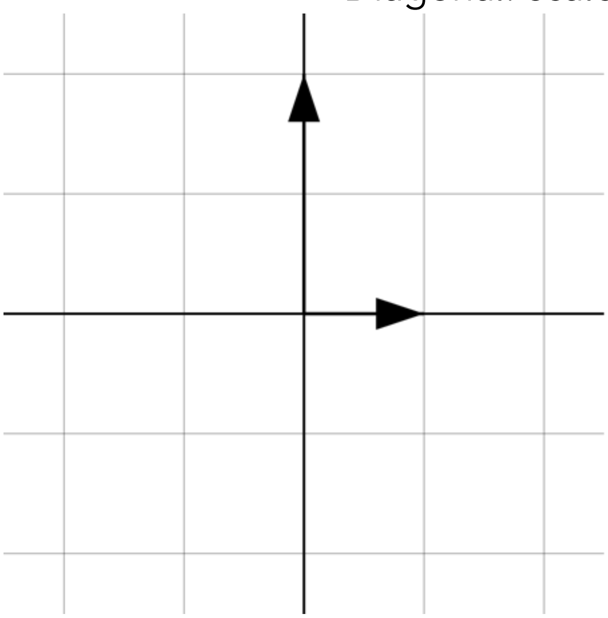
Orthonormal with det = −1

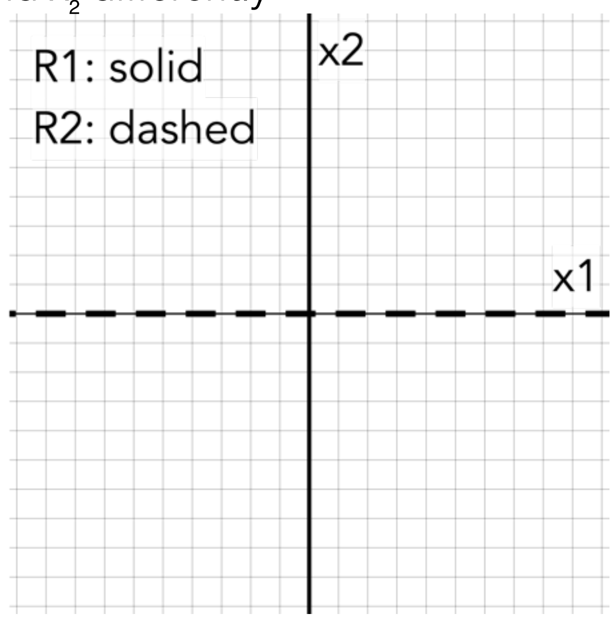




③ Non-uniform scaling = $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

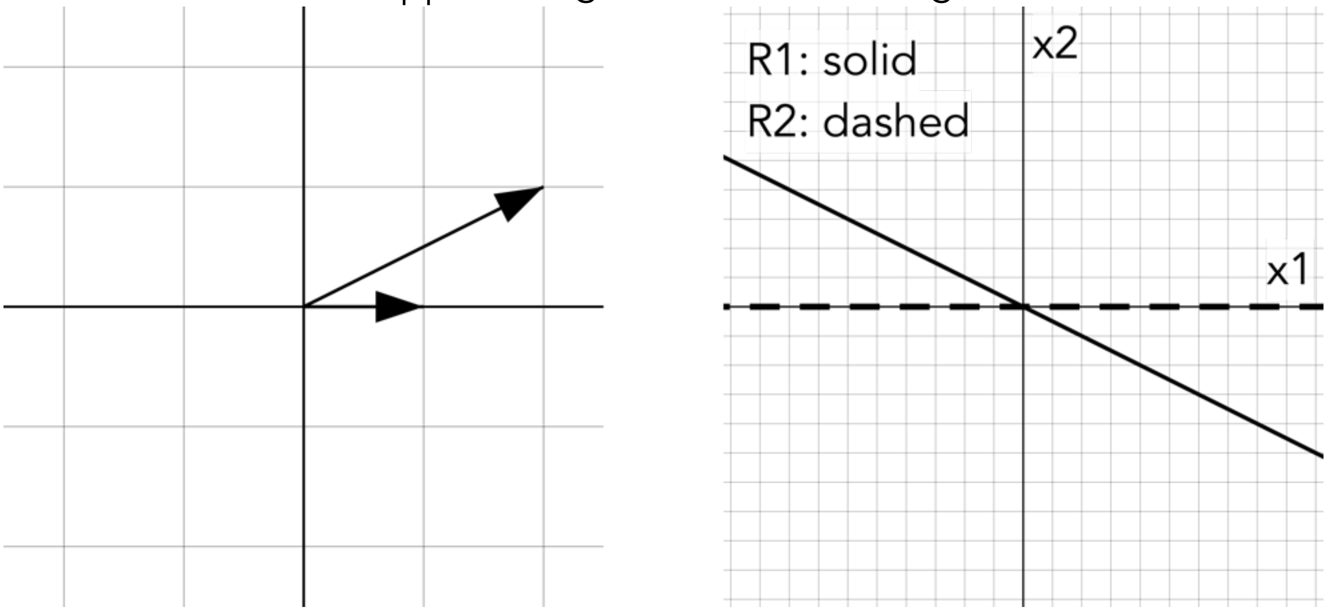
Diagonal: scales x_1 and x_2 differently





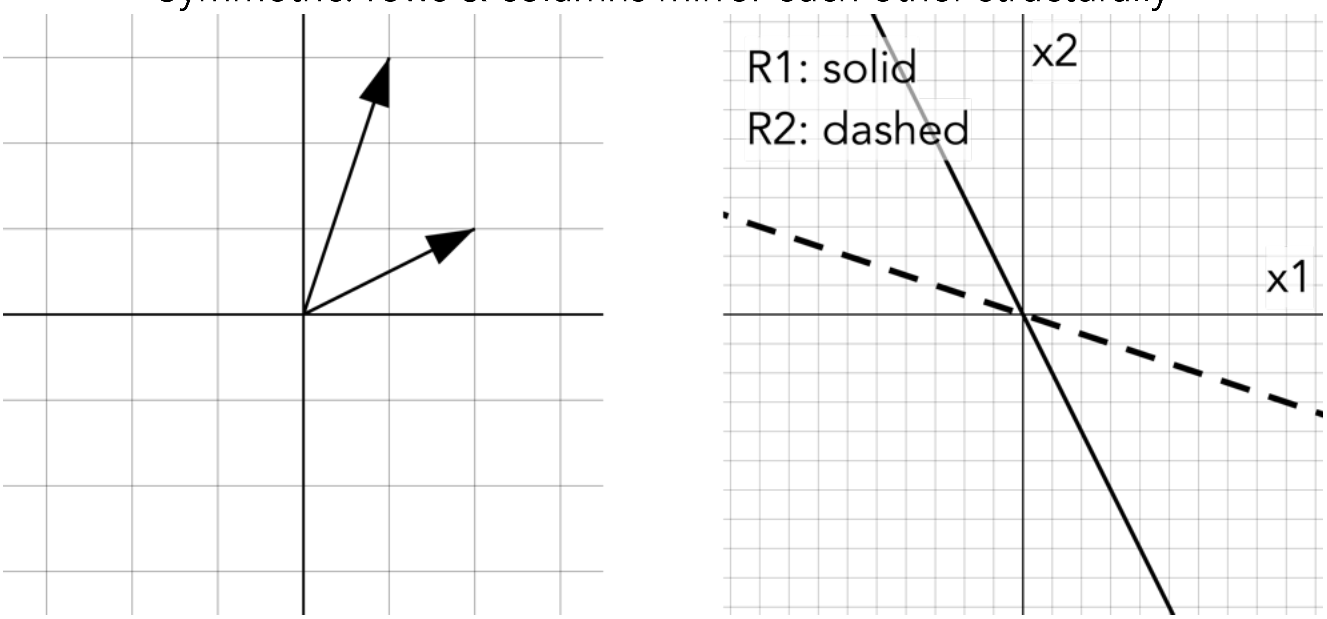
④ Shear = $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Upper-triangular: shear + scaling



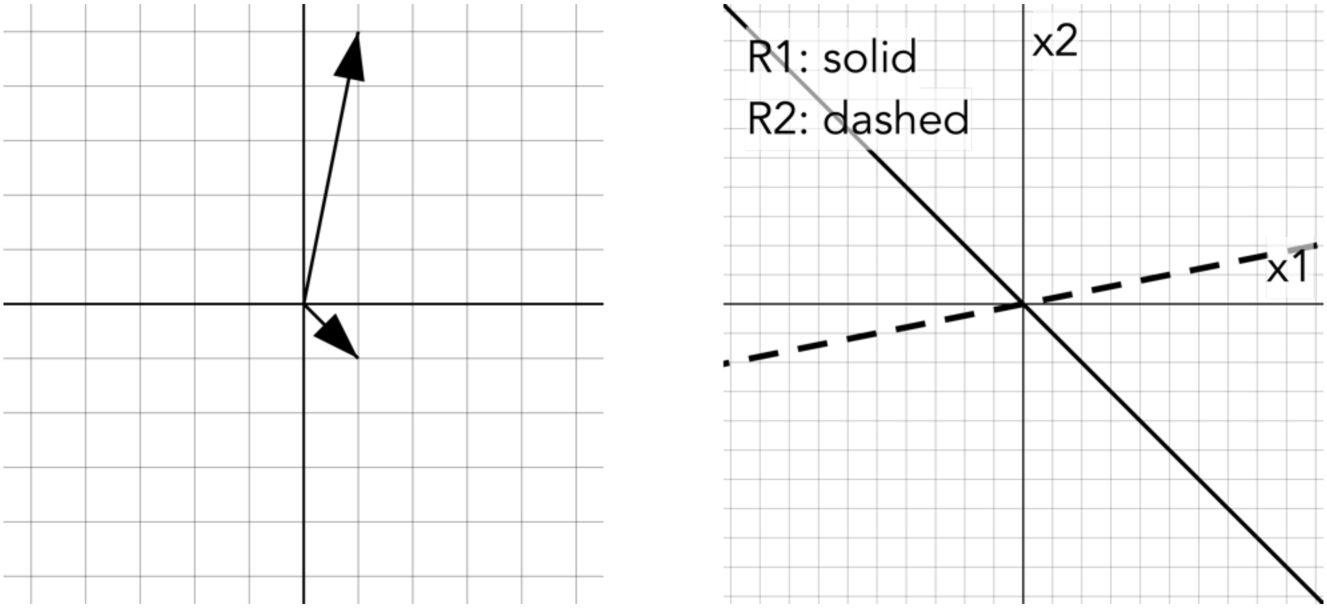
⑤ Symmetric full rank matrix = $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Symmetric: rows & columns mirror each other structurally



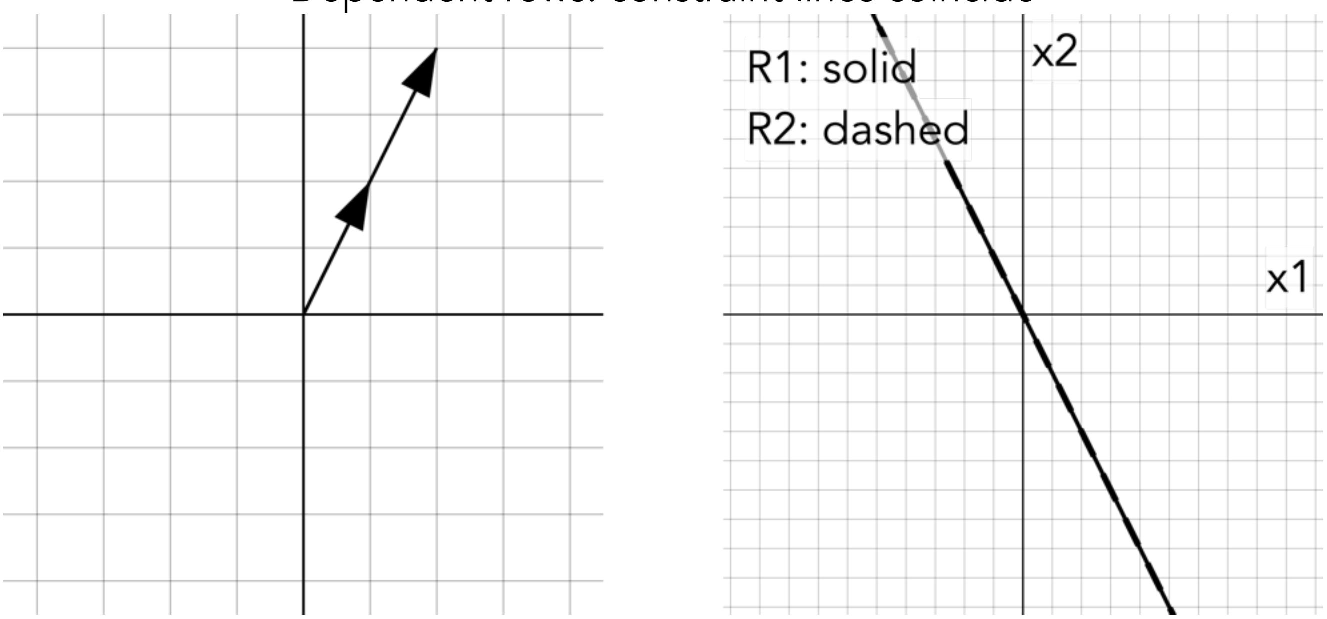
⑥ Random full rank matrix = $\begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$

Generic case: two distinct constraint lines; intersection is {0}



⑦ Rank 1 matrix = $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

Dependent rows: constraint lines coincide



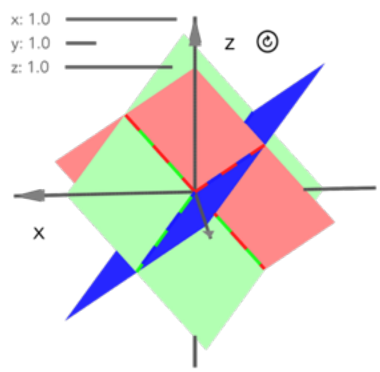
3D images of rows

In 2D, two independent rows define two distinct lines through the origin

In 3D, three independent rows define three planes
Each equation $\vec{r}_i \cdot \vec{x} = 0$ defines a plane through the origin
$(ax + by + cz = d = 0)$
Each such plane is orthogonal to its corresponding row vector $\vec{r}_i$

When the three rows are independent (rank = 3),
the only common intersection of the three planes is the origin

This is the 3×3 full-rank equivalent of the 2×2 case:



Definitions

We will now formalize the same geometric structures

① Vector space $\mathcal{V}$ or $\mathbb{R}^n$:

collection of all vectors with n components

Example: $\mathbb{R}^3$

② Subspace $\mathcal{S}$ of $\mathcal{V}$:

collection of all vectors satisfying the following:

- for any vectors $\vec{x}$ & $\vec{y}$ inside $\mathcal{S}$, all of their linear combinations are also inside $\mathcal{S}$ (which also implies that $\mathcal{S}$ includes the zero vector)

Examples: plane, line or zero vector in $\mathbb{R}^3$ (through the origin)

- Connection between an subspace and a span of set of vectors:
  - Span of vectors $\vec{a}_1 \ldots \vec{a}_n$ is a subspace
  - Any subspace can be spanned by a miminal number of vectors $\vec{a}_i$

③ Complementary subspaces $\mathcal{S}1$ & $\mathcal{S}2$ of $\mathcal{V}$ satisfy the following conditions:

- intersect only at 0
- every vector $\vec{x}$ in $\mathcal{V}$ can be written uniquely as $\vec{x} = \vec{x}1 + \vec{x}2$ with $\vec{x}1$ in $\mathcal{S}1$ and $\vec{x}2$ in $\mathcal{S}2$

or, equivalently, $\mathcal{S}1$ & $\mathcal{S}2$ span $\mathcal{V}$

④ Orthogonal subspaces $\mathcal{S}1$ & $\mathcal{S}2$ satisfy the following condition:

every vector in $\mathcal{S}1$ is orthogonal to every vector in $\mathcal{S}2$

Note:

- Complementary does not imply orthogonal
- Orthogonal does not imply complementary ($\mathcal{S}1$ & $\mathcal{S}2$ may not span $\mathcal{V}$)
- $\mathcal{S}2$ as the set of ALL vectors orthogonal to all vectors in $\mathcal{S}1$ does imply complementary ($\mathcal{S}1$ & $\mathcal{S}2$ span $\mathcal{V}$)

• If $\mathcal{S}1$ & $\mathcal{S}2$ are complementary,
most vectors do not lie entirely in $\mathcal{S}1$ or $\mathcal{S}2$:
they have components in both $\mathcal{S}1$ & $\mathcal{S}2$

• Complementary is a dimension condition:
$\dim(\mathcal{S}1) + \dim(\mathcal{S}2) = \dim(\mathcal{V})$ and $\mathcal{S}1 \cap \mathcal{S}2 = \{0\}$