Definition of echelon form

Pivot: first non-zero entry of a non-zero row

Row echelon form requirements:
1. Each pivot is to the right of the pivot in the row above
2. Rows containing only zeros appear on the bottom

Above requirements create a 'staircase' pattern
where each pivot has only zero entries
- in its row to the left
- in its column below

for row echelon form, pivots are not required to be normalized

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & m_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot can be viewed as 'declaration of independence' 🗽
Since each pivot has only zeros in its row to the left,
it cannot be written as linear combination of preceding columns

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Definition of reduced echelon form

Pivot: first non-zero entry of a non-zero row

Reduced echelon form requirements:

1. Each pivot is to the right of the pivot in the row above (same as for echelon)
2. Rows containing only zeros appear on the bottom (same as for echelon)
3. All entries above pivots are 0 (new requirement)
4. Pivots are normalized (new requirement)

Each pivot has only zero entries

- in its row to the left
- in its column below
- in its column above

$$\begin{bmatrix} 1 & 0 & m_{13} & 0 \\ 0 & 1 & m_{23} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Utility of row reduction algorithm

- Solving systems of linear equations

When a matrix represents a system of linear equations,
row reduction transforms the system into an equivalent one that is easy to solve

- pivot columns correspond to leading variables

‣ non-pivot columns correspond to free variables
(details are covered under 'Linear equations')

- Gaining structural understanding of matrices
  ‣ rank
  ‣ existence and location of pivot columns

- More: algorithms for computing
  ‣ matrix inverse
  ‣ matrix determinant
  ‣ QR factorization via Gram–Schmidt algorithm
  ‣ eigenvectors from eigenvalues (via solving equation systems)

Row reduction algorithm overview

Row reduction algorithm is used to
transform matrices into echelon or reduced echelon form
using 3 types of elementary operations:

- Row swap $R_i \Leftrightarrow R_j$
  is used if current pivot candidate is 0
- Row replacement $R_j \leftarrow R_j + sR_i$
  adds multiple of source Row i to target Row j
  to create zeros in pivot column above or below pivot
  ‣ below pivot (reduced echelon & echelon form)
  ‣ above pivot (reduced echelon form)

- Row scaling $R_i \leftarrow sR_i$ ($s \neq 0$)

scales pivot rows so that each pivot becomes 1
(required for reduced echelon transformation)

When are row exchanges needed?

Row exchange $R_i \Leftrightarrow R_j$ is done when
- current pivot candidate is zero
- nonzero entry exists below it in the same column

Suppose 4×4 matrix M has full rank,
and 2×2 left upper submatrix A has rank 1

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

After applying row reduction within the leading 2×2 block,
the second row of the block becomes zero,
so no valid pivot exists in that position

Elementary matrices

Performing a row operation is equivalent to
left-multiplying by an elementary matrix E

⇩

Transforming matrix M to echelon matrix R
is equivalent to

$R = E(k) \times \ldots \times E(1) \times M$

‣ k = number of row reduction steps performed

⦿ Elementary matrix of row swap or permutation E(p)
swaps row i and row j and is obtained
by swapping rows i and j of identity matrix I
(matrix shown below swaps rows 1 & 2)

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⦿ Elementary matrix of row replacement E(r)
adds multiple of row i to row j and is obtained
by adding multiple of row i to row j in identity matrix I
(matrix shown below adds s times row 1 to row 2)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ s & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⦿ Elementary matrix of row scaling $E(s)$
scales row i and is obtained
by scaling row i of identity matrix I
(matrix shown below scales row 1 by s)

$$\begin{bmatrix} s & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Vector span

Vector span describes the space generated by the vectors
Given original vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$,
their span is a set of all vectors
that can be built from the original set:

$$s_1\vec{v}_1+s_2\vec{v}_2+\ldots+s_n\vec{v}_n$$

or all linear combinations of original vectors

Geometric intuition:

- one vector: line through origin
- collinear or linearly dependent vectors: same line
- not collinear or linearly independent vectors: plane through origin

Linear independence & linear dependence

Set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

• is linearly independent if

each vector contributes a new direction

⇔

removing any one vector reduces the span

• is linearly dependent if

at least one of the vectors does not contribute a new direction

⇔

removing that vector does not change the span

Algebraic equivalents:

• Linearly independent:

$s_1\vec{v}_1+s_2\vec{v}_2+\ldots+s_n\vec{v}_n = \vec{0}$

implies all coefficients s = 0

• Linearly dependent:

$s_1\vec{v}_1+s_2\vec{v}_2+\ldots+s_n\vec{v}_n = \vec{0}$

there exists a non-zero choice of coefficients (not all s are 0)

Definition & calculation of rank

Rank of a matrix is the number of linearly independent rows or, equivalently, columns

- Row rank = number of independent rows
- Column rank = number of independent columns

Geometric meaning:

- Rank 1: all vectors ∈ one line
- Rank 2: all vectors ∈ one plane
- Rank 3: the vectors span $\mathbb{R}^3$

and similarly for higher dimensions

Rank is calculated as number of pivot rows or pivot columns of corresponding echelon matrix

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Row reduction steps preserve rank

- Row swap $R_i \Leftrightarrow R_j$

reorders rows without changing linear dependence

- Row scaling $R_i \leftarrow sR_i$

scales row by nonzero constant
without changing linear dependence

- Row replacement $R_j \leftarrow R_j + sR_i$

creates a new row in the span of pre-existing rows

Row rank = Col rank ⇔ rank(M) = rank(M^T)

Row reduction steps preserve rank

⇓

Rank of matrix can be obtained from its echelon form

From echelon matrix definition:

- each nonzero row contains one pivot: the first nonzero entry in that row
- each pivot lies in a distinct column

⇓

number of pivot rows = number of pivot columns

⇓

row rank = column rank

Example below shows a 4×4 rank-3 matrix with

- 3 pivot columns (1, 2, 4)
- 3 pivot rows (1, 2, 3)

m_{11}	m_{12}	m_{13}	m_{14}
0	m_{22}	m_{23}	m_{24}
0	0	0	m_{34}
0	0	0	0

Since rows of M are columns of M^T and columns of M^T are rows of M,
rank(M) = rank(M^T)

Rows & columns before & after row reduction

• Effect of row operations on rows:

Rows are replaced by linear combinations of existing rows

⇓

Resulting echelon matrix has

‣ r nonzero rows, each containing a pivot (r=rank)

‣ n-r zero rows, placed at the bottom

nonzero rows represent r independent directions determined by the original rows

• Effect of row operations on columns:

Column operations are not performed

⇓

columns of original matrix correspond

position-by-position to columns of echelon matrix

‣ linearly independent columns → pivot columns

‣ zero columns → zero columns

‣ linearly dependent columns → non-pivot columns

Reduced row echelon form of a full-rank n × n matrix is the Identity matrix

From previous page:

linearly independent columns produce pivot columns
in reduced echelon (and echelon) form

⇩

every column of reduced echelon matrix contains a pivot

From that and 'staircase' definition of reduced echelon (and echelon) forms follows
that resulting matrix must be of following type:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Proof of linearly dependent columns → non-pivot columns & the corollary

① Proof of linearly dependent columns →

non-pivot columns
(follows from linearity of matrix multiplication)

Suppose rank-3 matrix M consists of
- linearly independent vectors m1 & m2
- linearly dependent m3 = c1m1+ c2m2

‣ linearly independent m4

$$M = \left[m1, m2, (c1m1+c2m2), m4\right]$$

Each row reduction step is equivalent to left-multiplication by an elementary matrix

⇩

Row reduction is equivalent to left-multiplication by their product denoted as E:

$$R = EM$$

⇔

$$R = E\left[m1, m2, (c1m1+c2m2), m4\right]$$

From linear property of matrix multiplication:

$$R = E\left[m1, m2, (c1m1+c2m2), m4\right] = \left[Em1, Em2, E(c1m1+c2m2), Em4\right] =$$

$$\left[Em1, Em2, (Ec1m1+Ec2m2), Em4\right] = \left[Em1, Em2, (c1Em1+c2Em2), Em4\right]$$

⇩

in R, column 3 is also a linear combination of columns 1 & 2

Recall 'staircase' structure of echelon or reduced echelon:

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & m_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

any pivot has only zeros to the left in its row, or, equivalently,
no pivot column can be presented as linear combination of preceding ones
Thus, column 3 is a non-pivot column

Same logic applies to zero columns: $E0 = 0$

② Proof of independent columns in M →

independent columns in R:
(using column 4 as example)

Suppose columns m1, m2, and m4 are independent in M
This means $c1m1 + c2m2 + c4m4 = 0$ only if $c1=c2=c4=0$

In the reduced matrix $R = EM$:

$$c1r1 + c2r2 + c4r4 = c1(Em1) + c2(Em2) + c4(Em4)$$

$$\Downarrow$$

$$E(c1m1 + c2m2 + c4m4) = 0$$

Since E is invertible, it cannot map a non-zero vector to 0
Therefore, the only solution in R is also $c1=c2=c4=0$

Thus, independent columns in M produce independent columns in R
In our example, m4 (col 4) 'declares independence' 🗽
by providing the 3rd pivot in row 3

What echelon form reveals

Echelon form is non-unique and reveals structure of matrix or equation system:

- Consistency of system
(same as presence of solution)

for instance, system that row reduces to
matrix shown below (with pivot in the constant column)
has no solution

$$\left[\begin{array}{ccc|c} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & m_{34} \neq 0 \end{array}\right]$$

- Free variables correspond to non-pivot columns (red)
- Constrained variables correspond to pivot columns (green)

$$\left[\begin{array}{ccc|c} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Echelon form does not isolate the variables or present a solution

To obtain explicit solutions,
we continue row reduction to reduced echelon form
see 'Linear equations' on how to obtain solutions
from reduced echelon matrix

Partial pivoting

Strategy used as part of row reduction algorithm:

Every time a pivot is used to eliminate entries in its column,

- entry with largest absolute value in that column is selected
- If necessary, a row swap is performed so that this entry becomes the pivot

This reduces

- numerical error associated with division by small numbers
- risk of numerical overflow during computation

The algorithm demonstrated here is optimized for teaching and clarity over numerical stability and does not utilize this strategy