Reflection across a subspace $S \subset \mathbb{R}^n$

$\mathbb{R}^n = S \oplus S\perp$

($S\perp$ is orthogonal complement of S)

↓

Any vector $\vec{v} \in \mathbb{R}^n$ can be decomposed into 2 components

- $\vec{v}\| \in S$
- $\vec{v}\perp \in S\perp$

Reflection across S is a function that

- preserves $\vec{v}\|$ (S is the fixed subspace)
- negates $\vec{v}\perp$ ($S\perp$ is the negated subspace)

Will denote

- P: orthogonal projection matrix onto S
- R: reflection matrix
- $\vec{w}$: reflected vector R $\vec{v}$

$P\,\vec{v} = \vec{v}\|$

↓

$\vec{w} = \vec{v} - 2\vec{v}\perp = \vec{v}\| - \vec{v}\perp$

$= P\,\vec{v} - \vec{v}\perp = P\,\vec{v} - (\vec{v} - P\,\vec{v})$

$= (2\,P - I)\,\vec{v}$

↓

$R = 2\,P - I$

Applying reflection matrix R twice maps $\vec{v}$ back to $\vec{v}$,
which can be shown algebraically:

$R\,\vec{v} = R\,(\vec{v}\| + \vec{v}\perp)$

$= \vec{v}\| - \vec{v}\perp$

$= \vec{w}$

$R\vec{w} = R(\vec{v}_{\parallel} - \vec{v}_{\perp})$
$= R\vec{v}_{\parallel} - R\vec{v}_{\perp}$

- $R\vec{v}_{\parallel} = \vec{v}_{\parallel}$
- $R\vec{v}_{\perp} = -\vec{v}_{\perp}$

↓

$R\vec{w} = \vec{v}_{\parallel} - (-\vec{v}_{\perp})$
$= \vec{v}_{\parallel} + \vec{v}_{\perp}$
$= \vec{v}$

↓

$R(R\vec{v}) = \vec{v}$

↓

$R^2\vec{v} = \vec{v}$

↓

$R^2 = I$

↓

R is its own inverse

↓

R is always full rank

Edge cases:

① Reflection across $S = \{0\}$:

$\vec{v}_{\parallel} = \vec{0}$

↓

$\vec{v}_{\perp} = \vec{v}$

↓

$\vec{w} = -\vec{v}$

↓

$R = -I$

② Reflection across $S = \mathbb{R}^n$:

$\vec{v}_{\perp} = \vec{0}$

↓

$\vec{w} = \vec{v}$

↓

R = I

Reflection across a line spanned by $\vec{u} \subset \mathbb{R}^3$

Let S be the line we reflect across

S = span{ $\vec{u}$ }

where $\vec{u} \neq 0$

U = [$\vec{u}$]

↓

P (orthogonal projection onto the line) =

$U (U^T U)^{-1} U^T$

$$= \frac{\vec{u}\,\vec{u}^T}{\vec{u}^T\,\vec{u}}$$

↓

R (reflection across S) = 2P - I

$$= 2 \frac{\vec{u}\,\vec{u}^T}{\vec{u}^T\,\vec{u}} - I$$

Reflection across a line in $\mathbb{R}^3$:

- preserves 1 direction
- negates 2 directions

↓

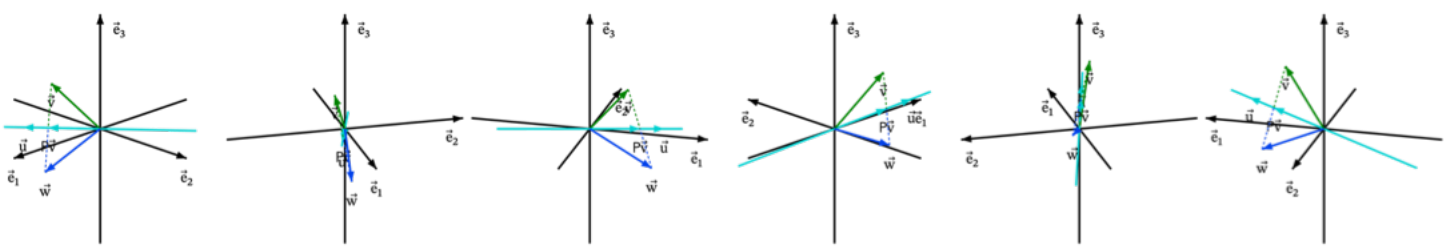
$$\det(R) = 1$$

Images show reflection across the line defined by

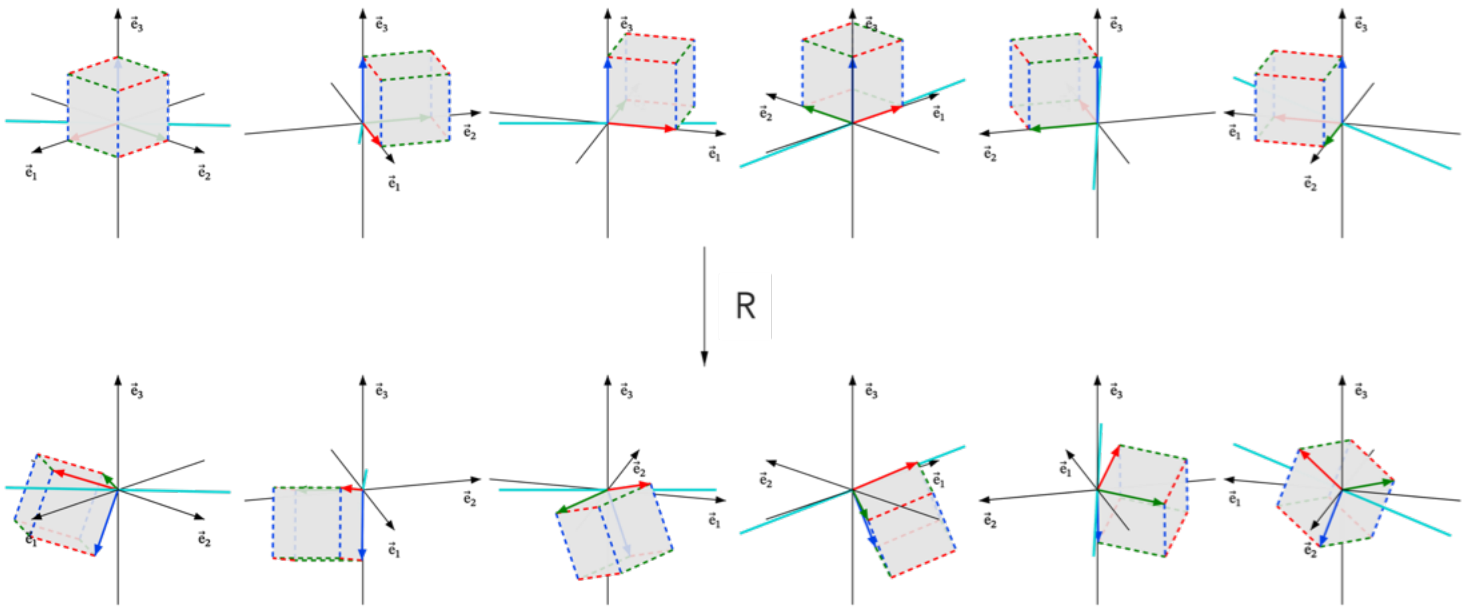
$$U = [\,\vec{u}\,] = \begin{bmatrix} 4 \\ -1.2 \\ 0.8 \end{bmatrix}$$

$$\text{reflection matrix } R = 2\,\vec{u}\,\frac{\vec{u}^{T}}{\vec{u}^{T}\,\vec{u}} - I = \begin{bmatrix} \approx 0.77 & \approx -0.531 & \approx 0.354 \\ \approx -0.531 & \approx -0.841 & \approx -0.106 \\ \approx 0.354 & \approx -0.106 & \approx -0.929 \end{bmatrix}$$

① Reflection of vector $\vec{v} = \begin{bmatrix} 2.2 \\ -1.1 \\ 2.6 \end{bmatrix}$



② Reflection of identity cube $\begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix}$



Reflection across a plane $S \subset \mathbb{R}^3$

Let S be the plane we reflect across

S = span{ $\vec{u}_1$, $\vec{u}_2$ }
where $\vec{u}_1$ and $\vec{u}_2$ are linearly independent
$U = [\ \vec{u}_1\ \vec{u}_2\ ]$
↓
P (orthogonal projection onto the plane) = $U (U^T U)^{-1} U^T$
↓
R (reflection across S) = $2\, U (U^T U)^{-1} U^T - I$

Reflection across a plane in $\mathbb{R}^3$:

‣ preserves 2 directions

‣ negates 1 direction

↓

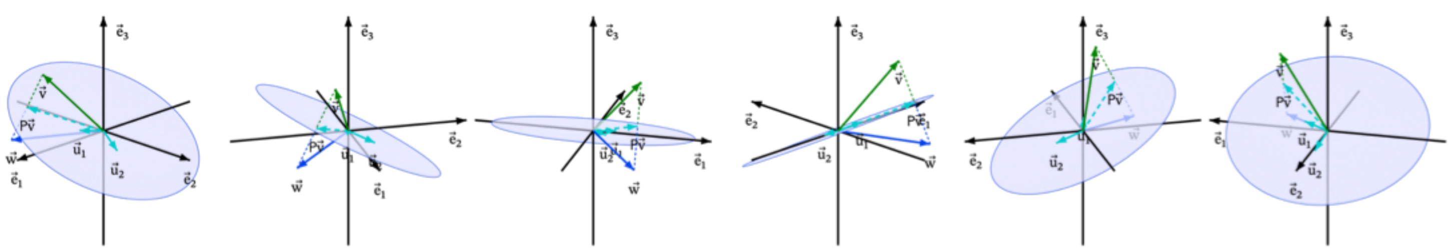
det(R) = -1

Images show reflection across the plane defined by

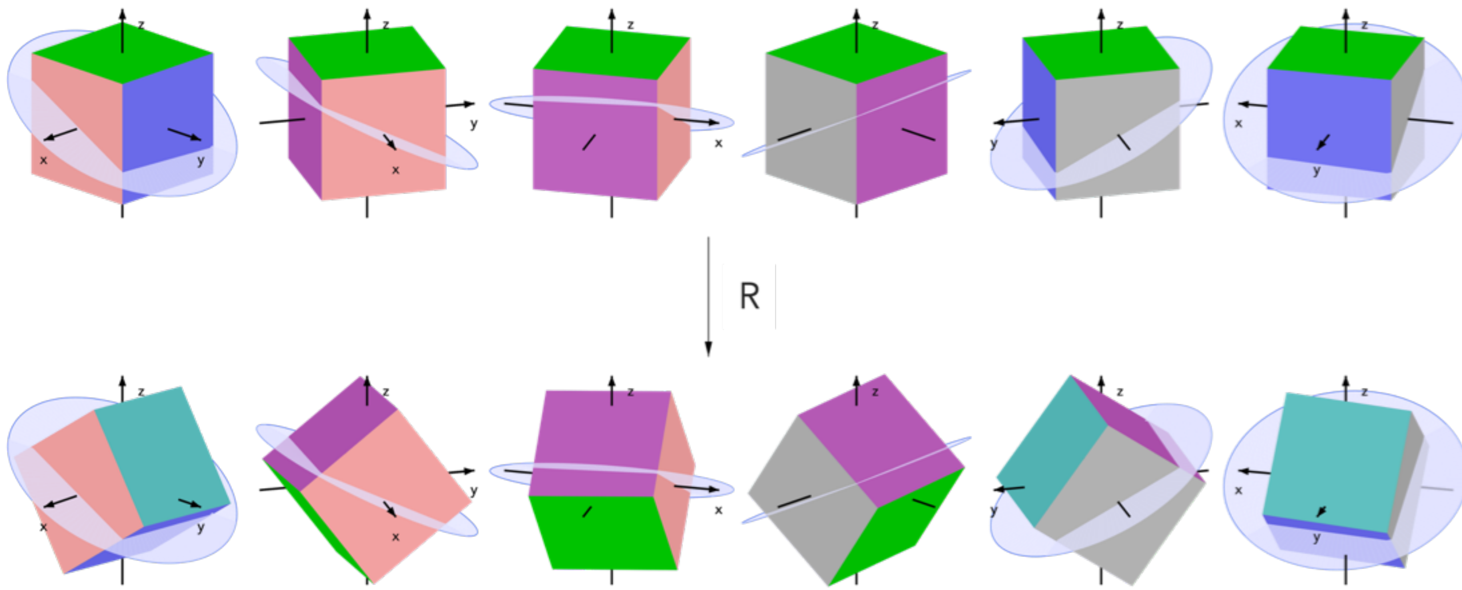
$$U = [\ \vec{u}_1 \mid \vec{u}_2\ ] = \begin{bmatrix} 1 & 0.25 \\ -0.3 & 1 \\ 0.2 & -0.5 \end{bmatrix}$$

$$\text{reflection matrix } R = 2\,U\,(U^T U)^{-1}\,U^T - I = \begin{bmatrix} \approx 0.997 & \approx 0.038 & \approx 0.074 \\ \approx 0.038 & \approx 0.586 & \approx -0.81 \\ \approx 0.074 & \approx -0.81 & \approx -0.582 \end{bmatrix}$$

① Reflection of vector $\vec{v} = \begin{bmatrix} 2.2 \\ -1.1 \\ 2.6 \end{bmatrix}$



② Reflection of identity cube $\begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix}$



The 2 conventions
(convention 1 is used in this tutorial)

	Convention 1	Convention 2
P projects onto	fixed subspace S	orthogonal complement $S\perp$
Projected component	$P\vec{v} = \vec{v}_{\parallel}$	$P\vec{v} = \vec{v}_{\perp}$
Reflection matrix	$R = 2P - I$	$R = I - 2P$
Equivalent notation	$R = 2P_S - I$	$R = I - 2P_{S\perp}$