Structure of matrix R

On the previous page, we demonstrated how to

- Take a full-column-rank matrix $A = [\ \vec{a}_1 \ldots \vec{a}_n\ ]$
  - Obtain matrix $Q = [\ \vec{e}_1 \ldots \vec{e}_r\ ]$ where
    - $\vec{e}_i \cdot \vec{e}_j = 0$ for $i \neq j$
    - $\vec{e}_i \cdot \vec{e}_i = 1$
  - $\vec{e}_j \cdot \vec{a}_i = 0$ for all $j > i$ (eq 1)
- $r$ = number of linearly independent columns in A
- from above, Q satisfies the condition $Q^T Q = I$

Next, will derive matrix $R = Q^T A$,
which comes from $A = Q R$

↓

$Q^T A = Q^T Q R$

$$\begin{bmatrix} \vec{e}_1^{\,T} \\ \vec{e}_2^{\,T} \\ \vec{e}_3^{\,T} \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} = \begin{bmatrix} \vec{e}_1^{\,T}\cdot\vec{a}_1 & \vec{e}_1^{\,T}\cdot\vec{a}_2 & \vec{e}_1^{\,T}\cdot\vec{a}_3 \\ \vec{e}_2^{\,T}\cdot\vec{a}_1 & \vec{e}_2^{\,T}\cdot\vec{a}_2 & \vec{e}_2^{\,T}\cdot\vec{a}_3 \\ \vec{e}_3^{\,T}\cdot\vec{a}_1 & \vec{e}_3^{\,T}\cdot\vec{a}_2 & \vec{e}_3^{\,T}\cdot\vec{a}_3 \end{bmatrix}$$

From equation 1, zeros appear everywhere below the diagonal:

$$\begin{bmatrix} \vec{e}_1^{\,T}\cdot\vec{a}_1 & \vec{e}_1^{\,T}\cdot\vec{a}_2 & \vec{e}_1^{\,T}\cdot\vec{a}_3 \\ \vec{e}_2^{\,T}\cdot\vec{a}_1 & \vec{e}_2^{\,T}\cdot\vec{a}_2 & \vec{e}_2^{\,T}\cdot\vec{a}_3 \\ \vec{e}_3^{\,T}\cdot\vec{a}_1 & \vec{e}_3^{\,T}\cdot\vec{a}_2 & \vec{e}_3^{\,T}\cdot\vec{a}_3 \end{bmatrix} = \begin{bmatrix} \vec{e}_1^{\,T}\cdot\vec{a}_1 & \vec{e}_1^{\,T}\cdot\vec{a}_2 & \vec{e}_1^{\,T}\cdot\vec{a}_3 \\ 0 & \vec{e}_2^{\,T}\cdot\vec{a}_2 & \vec{e}_2^{\,T}\cdot\vec{a}_3 \\ 0 & 0 & \vec{e}_3^{\,T}\cdot\vec{a}_3 \end{bmatrix}$$

$A = QR$ with orthonormal Q and upper-triangular R with the following dimensions:

- A: $m \times n$
- Q: $m \times r$

- R: $r \times n$

Diagonal entries of R

From the orthogonal decomposition

- $\vec{a}_i$ = (old directions) + $\vec{u}_i$
- $\vec{u}_i \perp$ (old directions)

only $\vec{u}_i$ contributes to the dot product with itself,
so $\vec{a}_i \cdot \vec{u}_i = \vec{u}_i \cdot \vec{u}_i > 0$

Since $\vec{e}_i$ is the normalized version of $\vec{u}_i$,
$r_{ii} = \vec{e}_i \cdot \vec{a}_i = \|\vec{u}_i\| \geq 0$

- $r_{ii} > 0$ if $\vec{a}_i$ adds a new independent direction
- $r_{ii} = 0$ if $\vec{a}_i$ lies in the span of previous columns

Looking ahead
In the QR iteration for eigenvalues,
each step factors A = QR and forms the next matrix as RQ
If diagonal entries of R were allowed to be negative,
columns of Q could flip sign from step to step,
causing artificial oscillations in the iteration

Requiring $r_{ii} \geq 0$ fixes the orientation of each orthogonal direction
and makes the QR iteration behave consistently

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Geometric view of QR factorization

(full column rank)

Consider matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad |\vec{a}_1| = \sqrt{2}$$

$$\downarrow$$

$$\vec{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\vec{a}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Projection of $\vec{a}_2$ on $\vec{e}_1$: $\text{proj} = (\vec{a}_2 \cdot \vec{e}_1)\, \vec{e}_1 = \left(\sqrt{2}\right) \vec{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{u}_2 = \vec{a}_2 - \text{proj} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$|\vec{u}_2| = \sqrt{2}$$

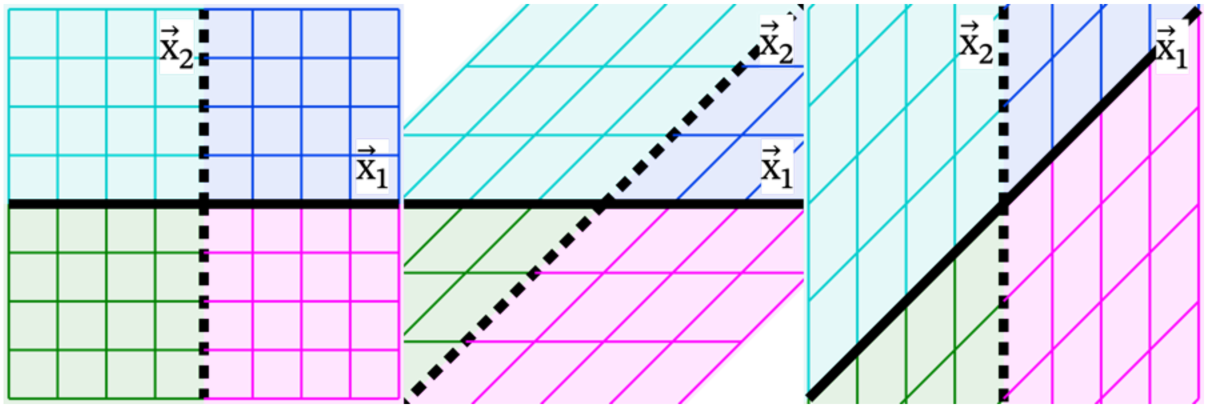
($\vec{u}_2$ is the component of $\vec{a}_2$ orthogonal to $\vec{e}_1$)

$$\vec{e}_2 = \frac{\vec{u}_2}{|\vec{u}_2|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

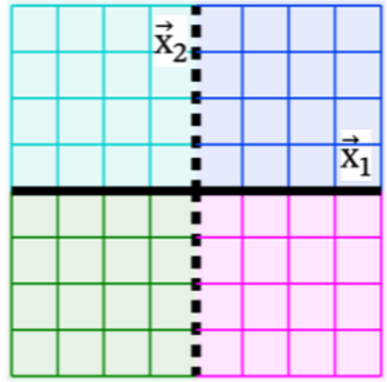
$$Q = [\, \vec{e}_1 \mid \vec{e}_2 \,] = \left[\begin{array}{c|c} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \hline \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right]$$

$$R = Q^T A = \left[\begin{array}{c|c} \sqrt{2} & \sqrt{2} \\ \hline 0 & \sqrt{2} \end{array}\right]$$

Now, as Q and R are derived, we can look at images
and see how this factorization
decouples deformation from rotation:
1. original grid
2. hierarchical distortion of the grid by R
(meaning of hierarchical will be discussed on the next page)
3. rotation or reflection of the deformed grid by Q

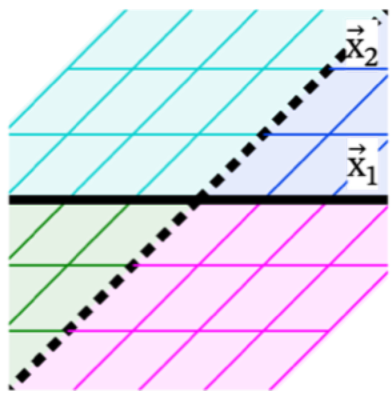

$\vec{x}_2$
$\vec{x}_1$
$\vec{x}_2$
$\vec{x}_1$
$\vec{x}_2$
$\vec{x}_1$

$$A = QR = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$



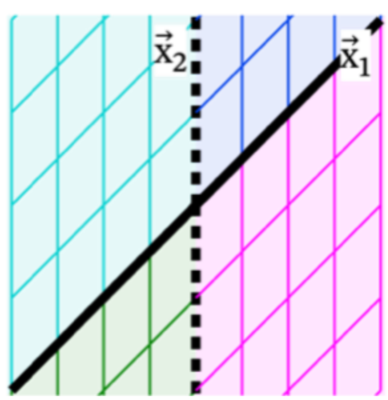

$$R = \begin{bmatrix} 1.414 & 1.414 \\ 0 & 1.414 \end{bmatrix}$$

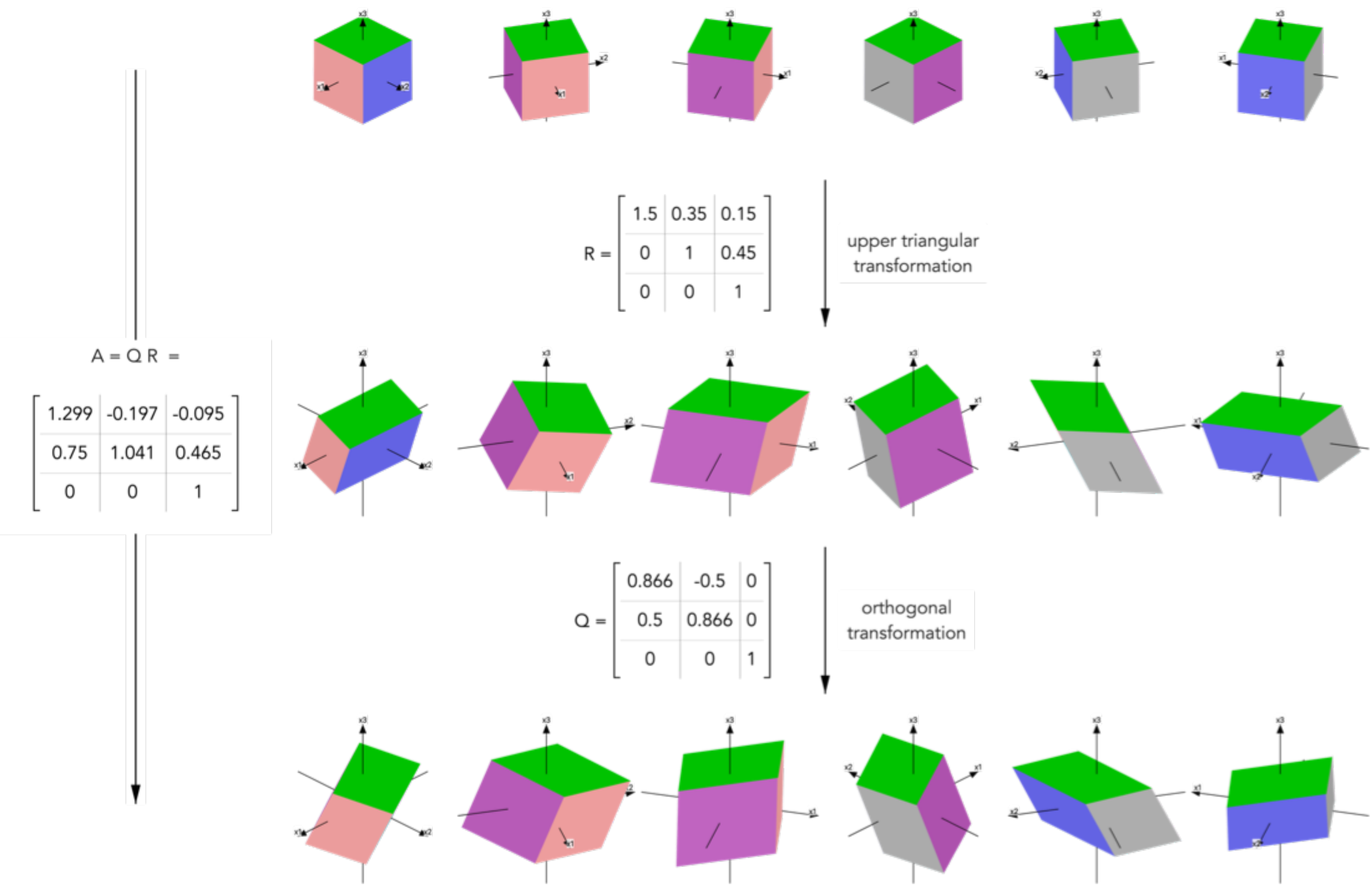
upper triangular transformation




$$Q = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}$$

orthogonal transformation




A = Q R =
1.299 -0.197 -0.095
0.75 1.041 0.465
0 0 1
R =
1.5 0.35 0.15
0 1 0.45
0 0 1
upper triangular transformation
Q =
0.866 -0.5 0
0.5 0.866 0
0 0 1
orthogonal transformation
x1
x2
x3

Alternative geometric view
(full column rank case)

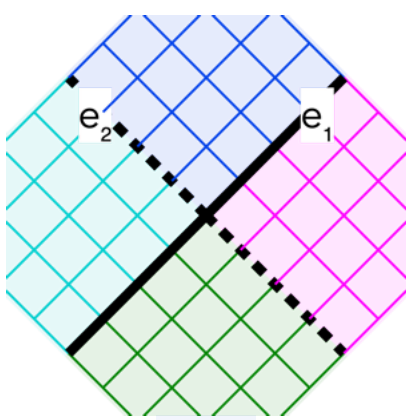
A = Q R admits two interpretations:
① Transformation sequence:
• deformation by R
• followed by rigid motion (rotation or reflection) by Q

② Basis interpretation:
• Q defines a new orthonormal basis
• R gives coordinates in this basis

For the same matrix shown on the previous page:

$$A = \left[\begin{array}{c|c} 1 & 0 \\ \hline 1 & 2 \end{array}\right] \qquad Q = [\,\vec{e}_1 \mid \vec{e}_2\,] = \left[\begin{array}{c|c} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \hline \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right]$$

Q defines the new orthonormal basis used for vectors R $\vec{x}$

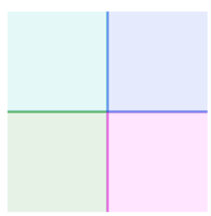


QRF of a 3×2 matrix

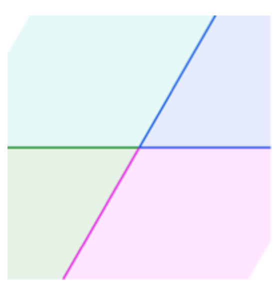
QR factorization of a rank-2 3×2 matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0.5\times\sqrt{\frac{2}{3}} \\ \frac{1}{2} & -0.5\times\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} 2 & \frac{1}{2} \\ 0 & \sqrt{\frac{3}{2}} \end{bmatrix}$$

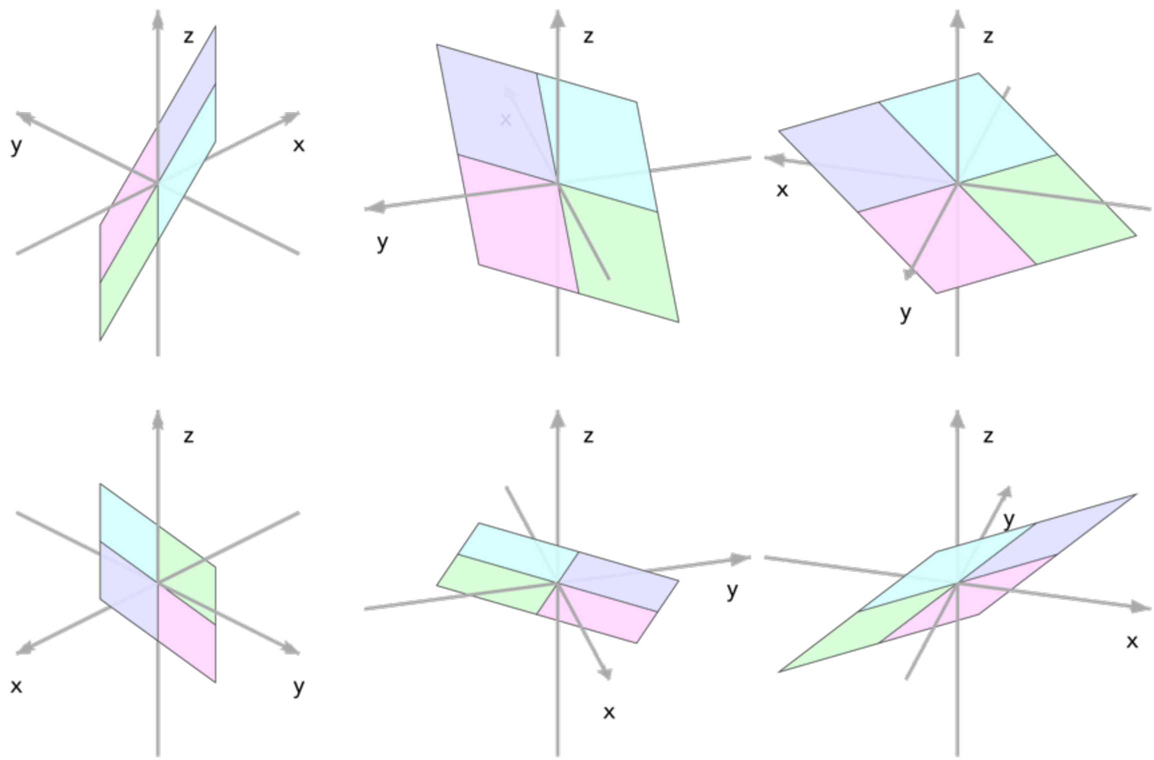
We visualize its domain $\mathbb{R}^2$ as a 4-color unit square



① The 2×2 upper-triangular matrix R deforms the square (shear followed by scaling)



② The 3×2 orthonormal matrix Q isometrically embeds
the deformed grid into $\mathbb{R}^3$,
so the image shows composite transformation by A = QR

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'Hierarchy' of R

Recall our derivation of R as

$$\begin{bmatrix} \vec{e}_1^{\mathsf{T}} \\ \vec{e}_2^{\mathsf{T}} \\ \vec{e}_3^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} = \begin{bmatrix} \vec{e}_1^{\mathsf{T}}\cdot\vec{a}_1 & \vec{e}_1^{\mathsf{T}}\cdot\vec{a}_2 & \vec{e}_1^{\mathsf{T}}\cdot\vec{a}_3 \\ \vec{e}_2^{\mathsf{T}}\cdot\vec{a}_1 & \vec{e}_2^{\mathsf{T}}\cdot\vec{a}_2 & \vec{e}_2^{\mathsf{T}}\cdot\vec{a}_3 \\ \vec{e}_3^{\mathsf{T}}\cdot\vec{a}_1 & \vec{e}_3^{\mathsf{T}}\cdot\vec{a}_2 & \vec{e}_3^{\mathsf{T}}\cdot\vec{a}_3 \end{bmatrix}$$

From equation 1, zeros appear everywhere below the diagonal:

$$\begin{bmatrix} \vec{e}_1^{\mathsf{T}}\cdot\vec{a}_1 & \vec{e}_1^{\mathsf{T}}\cdot\vec{a}_2 & \vec{e}_1^{\mathsf{T}}\cdot\vec{a}_3 \\ \vec{e}_2^{\mathsf{T}}\cdot\vec{a}_1 & \vec{e}_2^{\mathsf{T}}\cdot\vec{a}_2 & \vec{e}_2^{\mathsf{T}}\cdot\vec{a}_3 \\ \vec{e}_3^{\mathsf{T}}\cdot\vec{a}_1 & \vec{e}_3^{\mathsf{T}}\cdot\vec{a}_2 & \vec{e}_3^{\mathsf{T}}\cdot\vec{a}_3 \end{bmatrix} = \begin{bmatrix} \vec{e}_1^{\mathsf{T}}\cdot\vec{a}_1 & \vec{e}_1^{\mathsf{T}}\cdot\vec{a}_2 & \vec{e}_1^{\mathsf{T}}\cdot\vec{a}_3 \\ 0 & \vec{e}_2^{\mathsf{T}}\cdot\vec{a}_2 & \vec{e}_2^{\mathsf{T}}\cdot\vec{a}_3 \\ 0 & 0 & \vec{e}_3^{\mathsf{T}}\cdot\vec{a}_3 \end{bmatrix}$$

Upper triangular R encodes a hierarchy of dependencies
Each new output direction can only use the directions that came before it

For standard basis vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ we have
$R\,\vec{e}_1 \in \text{span}(\vec{e}_1)$
$R\,\vec{e}_2 \in \text{span}(\vec{e}_1, \vec{e}_2)$
$R\,\vec{e}_3 \in \text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$

Meaning of hierarchy as forward transformation:

- vector $\vec{e}_1$ scales only
- vector $\vec{e}_2$ shears towards $\vec{e}_1$ and scales
- vector $\vec{e}_3$ shears towards $\vec{e}_1$ & $\vec{e}_2$ and scales

Same hierarchy when solving $R\,\vec{x} = \vec{b}$ as backward transformation:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vec{x}_3 \end{bmatrix} = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix}$$

Note the order of dependence:

- forward for space transformation
- backward for solving

Uniqueness of QR factorization

Each orthogonal direction lies on a line with two possible orientations, so each column of Q could be flipped while preserving orthogonality

However, the Gram–Schmidt construction produces non-negative diagonal entries of R

This sign convention removes the ± ambiguity

Thin QR constructs an orthonormal basis for the column space of A and enforces an upper triangular R

Therefore, under these requirements, the thin QR factorization is unique even when A is rank-deficient