Concept:
zero all subdiagonal entries of each active column with one direct rotation.

Ⓐ

Input an $m\times n$ matrix A_1 and find an $m\times m$ matrix U_1 such that

- $U_1^{T} = U_1^{-1}$ with $\det\left(U_1\right) = 1$
- $A_2 = U_1 A_1$
- $\mathrm{col1}\left(A_2\right) = r_{1,1}\, \vec{e}_1$ where $r_{1,1}$ is a scalar
- $| r_{1,1} | = \ell_1 \quad \left(\text{length}\quad \left(\mathrm{col1}\left(A_1\right)\right)\right)$

In other words, U rotates A_1 so that its first column aligns with $\vec{e}_1$.

If $\mathrm{col1}\left(A_1\right)$ is already a multiple of $\vec{e}_1$,

$U_1 = I$

In this case, the diagonal entry $r_{1,1}$

of the resultant matrix A_2 may be positive, negative or 0.

Otherwise,

①

We define the rotation plane spanned by $\mathrm{col1}\left(A_1\right)$ and $\vec{e}_1$.

②

Construct the second unit vector $\hat{u}_2$ inside the plane such that

- $\hat{u}_2 \cdot \vec{e}_1 = 0$
- $| \hat{u}_2 | = 1$

$\hat{u}_2$ can be obtained by subtracting the projection of col1(A_1) onto $\vec{e}_1$ and normalizing the resultant vector

③

Define an orthonormal basis for the plane:

$$B_1 = [\, \vec{e}_1 \mid \hat{u}_2 \,]$$

④

Compute the angle θ between the direction of col1(A_1) and $\vec{e}_1$ using their normalized dot product

⑤

Extend the plane rotation to a rotation U_1 of $\mathbb{R}^m$

Denote J_1 as the 2×2 rotation by θ: $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$

in the coordinate system described by B_1.

U_1 performs the following geometric actions:

ⓐ leaves every vector perpendicular to the plane described by B_1 unchanged

ⓑ rotates every vector inside the plane by θ using $B_1 J_1 B_1^T$

Any vector $\vec{v}$ can be decomposed as

$$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}.$$

Since the columns of B_1 are orthonormal,

$$B_1^T B_1 = I_2.$$

The two components are

- $\vec{v}_{\parallel}$ is the projection onto the plane described by B_1:

$$\vec{v}_{\parallel} = B_1 B_1^T \vec{v}$$

- $\vec{v}_{\perp}$ is perpendicular to that plane:

$$\vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel}$$

$$= \left(I_m - B_1 B_1^T\right)\vec{v}$$

U_1 transforms these two components as follows:

- the component inside the plane is rotated:

$$\vec{v}'_{\parallel} = B_1 J_1 B_1^T \vec{v}$$

- the component perpendicular to the plane is unchanged:

$$\vec{v}'_{\perp} = \vec{v}_{\perp} = \left(I_m - B_1 B_1^T\right) \vec{v}$$

$$\downarrow$$

$$\vec{v}' = \vec{v}'_{\parallel} + \vec{v}'_{\perp} =$$

$$B_1 J_1 B_1^T \vec{v} + \left(I_m - B_1 B_1^T\right) \vec{v}$$

↓

$$U_1 = B_1 J_1 B_1^{\mathsf{T}} + \left(I_m - B_1 B_1^{\mathsf{T}}\right)$$

⑥

Simplify $U = B\,J\,B^{\mathsf{T}} + (I_m - B\,B^{\mathsf{T}})$:

$$U = I_m + B\,J\,B^{\mathsf{T}} - B\,B^{\mathsf{T}} =$$

$$= I_m + B\,(J - I_2)\,B^{\mathsf{T}}$$

Ⓑ

Apply U_1 to A_1:

$$A_2 = U_1 A_1.$$

The first column of A_2 is now complete:

$$\mathrm{col1}\left(A_2\right) = r_{1,1}\,\vec{e}_1.$$

↓

$$A_2 = \left[\begin{array}{c|c} r_{1,1} & \vec{*}^{\mathsf{T}} \\ \hline \vec{0} & \tilde{A}_2 \end{array}\right]$$

The first column has the required upper-triangular form.
It does not need to be changed again.

Deflate A_2 by setting aside its first row and first column.

The remaining active matrix is the trailing submatrix $\tilde{A}_2$, which has dimensions $(m - 1)\times(n - 1)$.

Repeat construction Ⓐ on $\tilde{A}_2$.

This produces an $(m - 1)\times(m - 1)$ rotation $\tilde{U}_2$

that transforms the first column of $\tilde{A}_2$ into a multiple of its first coordinate vector

Embed $\tilde{U}_2$ into an $m\times m$ matrix $U_2 = \left[\begin{array}{c|c} 1 & \vec{0}^T \\ \hline \vec{0} & \tilde{U}_2 \end{array}\right]$

Thus U_2 leaves the first row of A_2 unchanged and acts only on the deflated rows.

Set

$$A_3 = U_2 A_2.$$

Using the block forms of U_2 and A_2:

↓

$$A_3 = \left[\begin{array}{c|c} 1 & \vec{0}^T \\ \hline \vec{0} & \tilde{U}_2 \end{array}\right] \left[\begin{array}{c|c} r_{1,1} & \vec{*}^T \\ \hline \vec{0} & \tilde{A}_2 \end{array}\right] = \left[\begin{array}{c|c} r_{1,1} & \vec{*}^T \\ \hline \vec{0} & \tilde{U}_2 \tilde{A}_2 \end{array}\right]$$

By construction,

the first column of $\tilde{U}_2 \tilde{A}_2$ is a multiple of its first coordinate vector.

↓

Therefore, the first two columns of A_3 have upper-triangular form

Repeat this deflation and rotation process for each remaining active column

- The product of the transposed rotations forms an orthogonal matrix Q
- The final upper-triangular or upper-trapezoidal matrix R satisfies $A_1 = Q\,R$

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On this page, m denotes the number of active rows, and we omit the iteration subscripts.

For each column, the rotation matrix was derived as

$$U = I_m + B\,(J - I_2)\,B^{T}.$$

The following quantities have already been defined (not yet computed):

- $\ell = \text{length}\big(\text{col1}(A)\big)$
- $c = \cos(\theta)$
- $s = \sin(\theta)$
- $B = [\ \vec{e}_1 \mid \hat{u}_2\ ]$

Let

$$d = c - 1$$

and

$$D = J - I_2.$$

$$\downarrow$$

$$D = J - I_2 = \left[\begin{array}{c|c} d & s \\ \hline -s & d \end{array}\right]$$

$$\downarrow$$

$$U = I_m + B\,D\,B^T$$

$$B\,D = \left[\begin{array}{c|c} \vec{e}_1 & \hat{u}_2 \end{array}\right]\left[\begin{array}{c|c} d & s \\ \hline -s & d \end{array}\right] = \left[\begin{array}{c|c} d\,\vec{e}_1 - s\,\hat{u}_2 & s\,\vec{e}_1 + d\,\hat{u}_2 \end{array}\right]$$

$$B\,D\,B^T = \left[\begin{array}{c|c} d\,\vec{e}_1 - s\,\hat{u}_2 & s\,\vec{e}_1 + d\,\hat{u}_2 \end{array}\right]\left[\begin{array}{c} \vec{e}_1^{\,T} \\ \hat{u}_2^T \end{array}\right]$$

$$= d\,\vec{e}_1\,\vec{e}_1^{\,T} - s\,\hat{u}_2\,\vec{e}_1^{\,T} + s\,\vec{e}_1\,\hat{u}_2^T + d\,\hat{u}_2\,\hat{u}_2^T.$$

$$\downarrow$$

$$U = I_m + d\,\vec{e}_1\,\vec{e}_1^{\,T} - s\,\hat{u}_2\,\vec{e}_1^{\,T} + s\,\vec{e}_1\,\hat{u}_2^T + d\,\hat{u}_2\,\hat{u}_2^T.$$

Since the first entry of $\hat{u}_2$ is zero, $\hat{u}_2 = \begin{bmatrix} 0 \\ \bar{u}_2 \end{bmatrix}$

where $\bar{u}_2$ contains entries 2 through m of $\hat{u}_2$.

Using the block forms of $\vec{e}_1$ and $\hat{u}_2$, the four outer-product terms affect U as follows:

$$d\,\vec{e}_1\,\vec{e}_1^{\,T} = \left[\begin{array}{c|c} d & \vec{0}^T \\ \hline \vec{0} & 0_{m-1} \end{array}\right]$$

$$s\,\vec{e}_1\,\hat{u}_2^{\,T} = \left[\begin{array}{c|c} 0 & s\,\bar{u}_2^{\,T} \\ \hline \vec{0} & 0_{m-1} \end{array}\right]$$

$$-s\,\hat{u}_2\,\vec{e}_1^{\,T} = \left[\begin{array}{c|c} 0 & \vec{0}^T \\ \hline -s\,\bar{u}_2 & 0_{m-1} \end{array}\right]$$

$$d\,\hat{u}_2\,\hat{u}_2^{\,T} = \left[\begin{array}{c|c} 0 & \vec{0}^T \\ \hline \vec{0} & d\,\bar{u}_2\,\bar{u}_2^{\,T} \end{array}\right]$$

$$1 + d = c$$

$$\downarrow$$

$$U = I_m + d\,\vec{e}_1\,\vec{e}_1^{\,T} - s\,\hat{u}_2\,\vec{e}_1^{\,T} + s\,\vec{e}_1\,\hat{u}_2^{\,T} + d\,\hat{u}_2\,\hat{u}_2^{\,T} = \left[\begin{array}{c|c} c & s\,\bar{u}_2^{\,T} \\ \hline -s\,\bar{u}_2 & I_{m-1} + d\,\bar{u}_2\,\bar{u}_2^{\,T} \end{array}\right]$$

We can now summarize the quantities that must actually be computed.

Let $a = \big(\mathrm{col1}(A)\big)_1$, and let $\bar{p}$ contain entries 2 through m of col1(A). No projection computation is required.

- $\rho^2 = \bar{p}^T \bar{p}$
- $\ell = \sqrt{a^2 + \rho^2}$
- $\ell^{-1} = \dfrac{1}{\ell}$
- $c = a\ \ell^{-1}$
- $\bar{g} = \bar{p}\ \ell^{-1}$
- $h = \dfrac{c - 1}{\rho^2}$

For m active rows, computing these quantities requires

- 2m multiplications
- m additions and subtractions
- 2 divisions
- 1 square root.

Therefore, the block form becomes $U = \left[\begin{array}{c|c} c & \bar{g}^T \\ \hline -\bar{g} & I_{m-1} + h\ \bar{p}\ \bar{p}^T \end{array}\right]$

The quantities c, $\bar{g}$, and h determine the block matrix U.

However, instead of multiplying the quantities to compute U, we will derive the algebraic product U A

The block form derived on the preceding page contains two quantities that can be simplified before they are computed:

- $s\,\bar{u}_2$
- $d\,\bar{u}_2\,\bar{u}_2^T$.

Let $\bar{p}$ contain entries 2 through m of col1(A).
Since subtracting the projection onto $\vec{e}_1$ only removes the first entry of col1(A),

$$\vec{p} = \text{col1}(A) - \big(\text{col1}(A)\big)_1\,\vec{e}_1 = \begin{bmatrix} 0 \\ \bar{p} \end{bmatrix}$$

Let
$\rho = \text{length}(\vec{p})$.

Then

$$\bar{u}_2 = \frac{\bar{p}}{\rho}$$

and

$$s = \frac{\rho}{\ell}.$$

Simplify $s\,\bar{u}_2$:

$$s\,\bar{u}_2 = \left(\frac{\rho}{\ell}\right)\left(\frac{\bar{p}}{\rho}\right)$$

$$= \frac{\bar{p}}{\ell}.$$

Therefore, define

$$\bar{g} = \frac{\bar{p}}{\ell}.$$

Then
$s\,\bar{u}_2 = \bar{g}$,
so neither s nor ρ needs to be computed
for the off-diagonal blocks of U.

②
Simplify $d\,\bar{u}_2\,\bar{u}_2^{\mathsf{T}}$:

$$d\,\bar{u}_2\,\bar{u}_2^{\mathsf{T}}$$

$$= d\left(\frac{\bar{p}}{\rho}\right)\left(\frac{\bar{p}^{\mathsf{T}}}{\rho}\right)$$

$$= \left(\frac{d}{\rho^2}\right)\bar{p}\,\bar{p}^{\mathsf{T}}.$$

Only ρ^2 is required in this expression.
The square root $\rho = \text{length}(\vec{p})$ is not required.

Since

$\rho^2 = \bar{p}^T \bar{p}$,

the same sum of squares used in

$$\ell^2 = \big(\text{col1}(A)\big)_1^{\,2} + \rho^2$$

can be reused directly.

Thus, let

$$h = \frac{d}{\rho^2}.$$

The two simplified quantities are

$s\,\bar{u}_2 = \bar{g}$

and

$d\,\bar{u}_2\,\bar{u}_2^T = h\,\bar{p}\,\bar{p}^T$.

Therefore, U may be written as $\left[\begin{array}{c|c} c & \bar{g}^T \\ \hline -\bar{g} & I_{m-1} + h\,\bar{p}\,\bar{p}^T \end{array}\right]$

where

$$\bar{g} = \frac{\bar{p}}{\ell}$$

and

$$h = \frac{c-1}{\rho^2}.$$

The rotation can therefore be computed using only one square root:

$$\ell = \text{length}\big(\text{col1}(A)\big).$$

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Algebraic derivation of U A

This page concerns the general case $\rho^2 > 0$.
In the zero and parallel cases, $U = I_m$ and $U A = A$.

Separate the first column of A from its remaining columns:

$$A = \left[\ \text{col1}(A) \mid C\ \right]$$

The first column of U A is already known from the construction of U:
$$U\,\text{col1}(A) = \ell\,\vec{e}_1$$
↓
Only the remaining columns C must be transformed.

Partition C into its first row and lower block,
and write the first column of A using a and $\bar{p}$:
↓

$$C = \begin{bmatrix} \vec{b}^T \\ C_b \end{bmatrix} \qquad A = \left[\begin{array}{c|c} a & \vec{b}^T \\ \hline \bar{p} & C_b \end{array}\right]$$

$$\downarrow$$

$$U\,C = \left[\begin{array}{c|c} c & \bar{g}^T \\ \hline -\bar{g} & I_{m-1} + h\,\bar{p}\,\bar{p}^T \end{array}\right] \begin{bmatrix} \vec{b}^T \\ C_b \end{bmatrix} = \begin{bmatrix} c\,\vec{b}^T + \bar{g}^T\,C_b \\ -\bar{g}\,\vec{b}^T + C_b + h\,\bar{p}\,\bar{p}^T\,C_b \end{bmatrix}$$

The row vector $\bar{p}^T\,C_b$ is used in both blocks.

Compute it once:

$$\vec{z}^T = \bar{p}^T\,C_b$$

Since $\bar{g} = \bar{p}\,\ell^{-1}$,

$$\bar{g}^T\,C_b = \ell^{-1}\,\vec{z}^T.$$

$$\downarrow$$

$$U\,C = \begin{bmatrix} c\,\vec{b}^T + \ell^{-1}\,\vec{z}^T \\ -\bar{g}\,\vec{b}^T + C_b + h\,\bar{p}\,\vec{z}^T \end{bmatrix} = \begin{bmatrix} c\,\vec{b}^T + \ell^{-1}\,\vec{z}^T \\ C_b + (-\bar{g}\,\vec{b}^T + h\,\bar{p}\,\vec{z}^T) \end{bmatrix}$$

Combining the known first column with U C gives

$$U\,A = \left[\begin{array}{c|c} \ell & c\,\vec{b}^T + \ell^{-1}\,\vec{z}^T \\ \hline \vec{0} & C_b - \bar{g}\,\vec{b}^T + h\,\bar{p}\,\vec{z}^T \end{array}\right]$$

Thus U is never constructed.
The active matrix is updated using one row-vector product

and scaled outer-product terms.

Since $\bar{g} = \bar{p}\,\ell^{-1}$, define
$\vec{w}^{\mathsf{T}} = -\ell^{-1}\,\vec{b}^{\mathsf{T}} + h\,\vec{z}^{\mathsf{T}}$.

Then
$-\bar{g}\,\vec{b}^{\mathsf{T}} + h\,\bar{p}\,\vec{z}^{\mathsf{T}} = \bar{p}\,\vec{w}^{\mathsf{T}}$.
↓

$$U\,C = \begin{bmatrix} c\,\vec{b}^{\mathsf{T}} + \ell^{-1}\,\vec{z}^{\mathsf{T}} \\ C_b + \bar{p}\,\vec{w}^{\mathsf{T}} \end{bmatrix}$$

For an active m×n matrix, computing U C requires
- $(2m + 2)(n - 1)$ multiplications
- $(2m - 1)(n - 1)$ additions and subtractions

Including the quantities computed on the preceding page,
one complete active step requires
- $2m + (2m + 2)(n - 1)$ multiplications
- $m + (2m - 1)(n - 1)$ additions and subtractions
- 2 divisions
- 1 square root.

The first column of U A is now complete.
Deflate U A by setting aside its first row and first column,
and repeat the same construction on the trailing
$(m - 1)\times(n - 1)$ active matrix.

When no active column has entries below its diagonal position,
return the resulting upper-triangular or upper-trapezoidal matrix R
and the accumulated orthogonal matrix Q.

Computation cost: comparison with Givens and Householder methods
(Full explicit Q and R for an n×n matrix)

Method	Givens rotations	Direct plane rotations	Householder reflections
Multiplications	$\approx \left(\frac{10}{3}\right) n^3$	$\left(\frac{5}{3}\right) n^3 + \left(\frac{9}{2}\right) n^2 - \left(\frac{25}{6}\right) n - 2$	$\approx \left(\frac{4}{3}\right) n^3$
Additions and subtractions	$\approx \left(\frac{5}{3}\right) n^3$	$\left(\frac{5}{3}\right) n^3 - \left(\frac{2}{3}\right) n - 1$	$\approx \left(\frac{4}{3}\right) n^3$
Flops (multiplications + additions and subtractions)	$\approx 5n^3$	$\left(\frac{10}{3}\right) n^3 + \left(\frac{9}{2}\right) n^2 - \left(\frac{29}{6}\right) n - 3$	$\approx \left(\frac{8}{3}\right) n^3$
Divisions	$\approx n(n - 1)$	$2(n - 1)$	Implementation-dependent lower-order terms
Square roots	$\frac{n(n - 1)}{2}$	$n - 1$	$n - 1$
Weighted cost	$\approx 5n^3 + 12n(n - 1)$	$\left(\frac{10}{3}\right) n^3 + \left(\frac{9}{2}\right) n^2 + \left(\frac{115}{6}\right) n - 27$	$\geq \left(\frac{8}{3}\right) n^3 + 8(n - 1)$

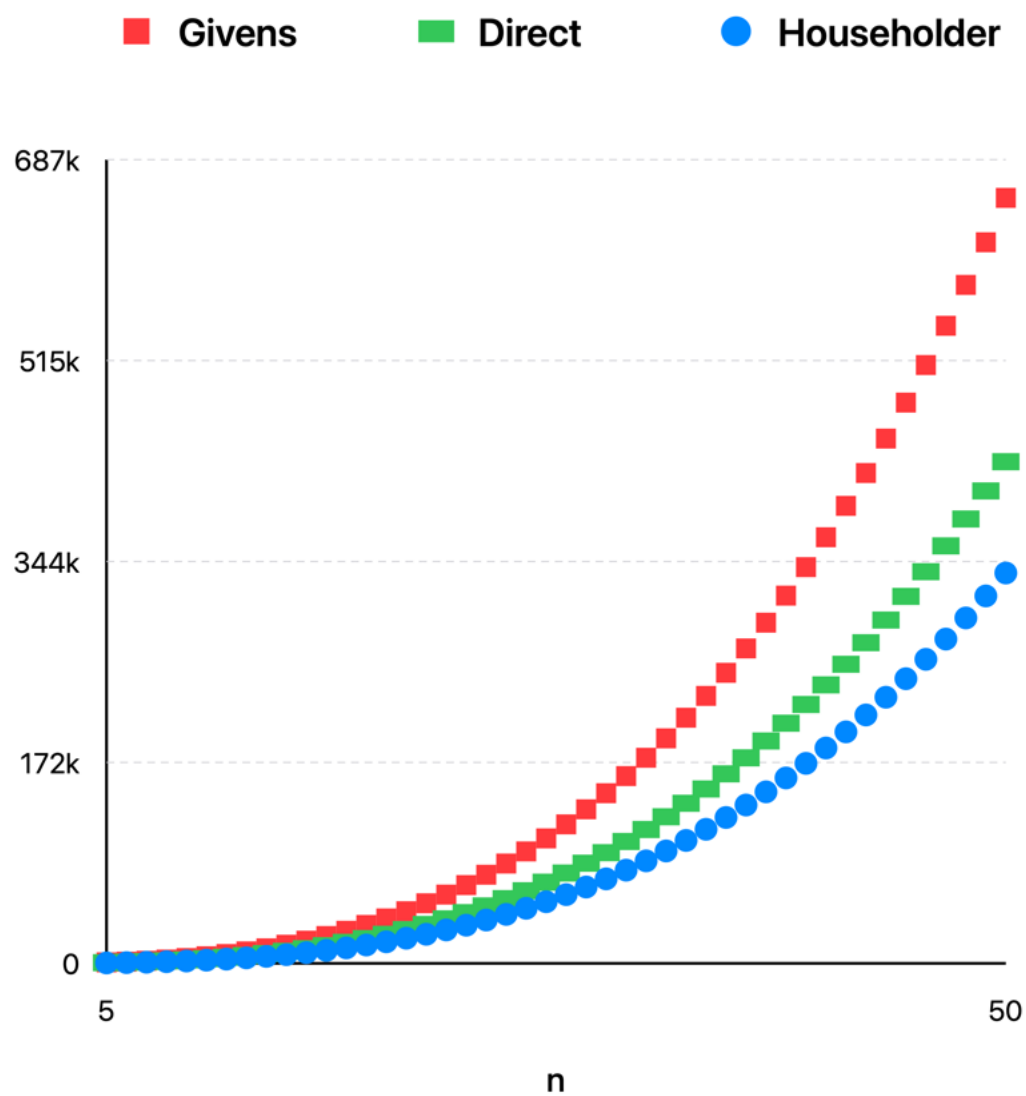
Blue: best cost among the three methods in that operation category.
Green: direct plane rotations improve on Givens rotations, but not on Householder.

Weighted-cost convention:

- additions, subtractions, and multiplications are assigned weight 1
- divisions and square roots are assigned weight 8

These weights are approximate and hardware-dependent.
The plot should be interpreted as a comparative estimate,
not as measured execution time.
The Householder value is a lower bound because its
implementation-dependent division count is not included.

The plot shows estimated weighted arithmetic cost for dense square matrices where
$5 \le n \le 50$



There may be significant opportunities to improve efficiency
by computing some of the quantities simultaneously.
We have not yet implemented these optimizations.

Open questions:
① How does numerical stability compare with Householder
when the target column is close to $\vec{e}_1$?
If we are lucky and the comparison is favorable,
a hybrid algorithm may be useful.

② How does efficiency compare with Givens rotations
for sparse matrices?
If we are lucky and the comparison is favorable,
a hybrid algorithm may also be useful.

③ Is there an advantage to a hybrid algorithm selecting
among Householder, Givens, and direct plane rotations
on a column-by-column basis?

④ Does this simplify computations for rank-1 QR updates?

⑤ Are there any other algebraic shortcuts that we have not implemented?

AI assistance:
AI tools were used extensively for information gathering,
checking references, and assisting with operation counts.
All mathematical claims, formulas, visualizations, and final
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