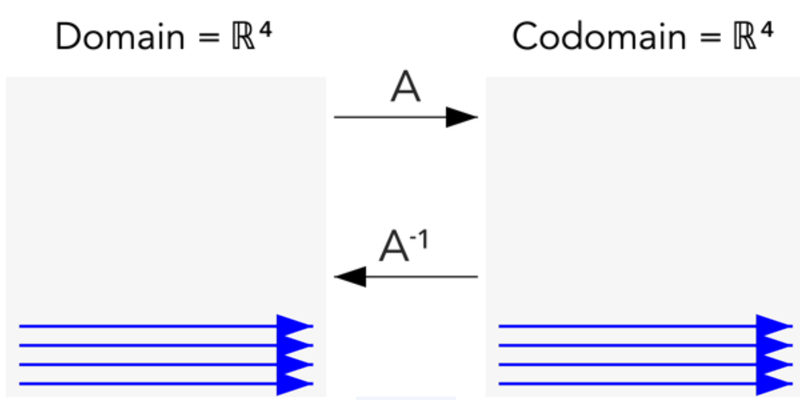
Simplest case: square full-rank A

A is a square full-rank (n=4)×(n=4) matrix
• dim(row space) = n
• dim(null space) = 0
↓
every domain vector is mapped to a unique codomain vector
• dim(col space) = n
• dim(ℓ-null space) = 0
↓
output(A) fills codomain

• every domain vector is mapped to a unique codomain vector by A
• every vector in the codomain has a unique vector in the domain
↓
there is a unique matrix A^{-1} that maps codomain vectors back to domain vectors



Tall full-column-rank A

A is a tall full-column-rank (m=6)×(n=4) matrix

• dim(row space) = n

• dim(null space) = 0

↓

every domain vector is mapped to a unique codomain vector

• dim(col space) = n

• dim(ℓ-null space) = m − n

↓

col space(A) does not fill codomain

• every domain vector is mapped to a unique codomain vector by A

• some vectors in the codomain are unreachable

• A is one-to-one but not onto

↓

there is no inverse matrix that maps all codomain vectors back to domain vectors

However, part of the transformation can be reversed

• every vector in col space(A) comes from a unique domain vector

• vectors in ℓ-null space(A) cannot be produced by A

So we can try to reverse A only on col space(A)

For this type of matrix,

$A^+ = (A^T A)^{-1} A^T$

is called the pseudoinverse of A

(note that $(A^T A)^{-1}$ is defined because A has full column rank)

A^+ acts as follows:

① maps vectors from col space(A) back to the corresponding vectors in the domain:

suppose $\vec{y} = A\,\vec{x}$

$A^+ \vec{y} = (A^T A)^{-1} A^T \vec{y} =$

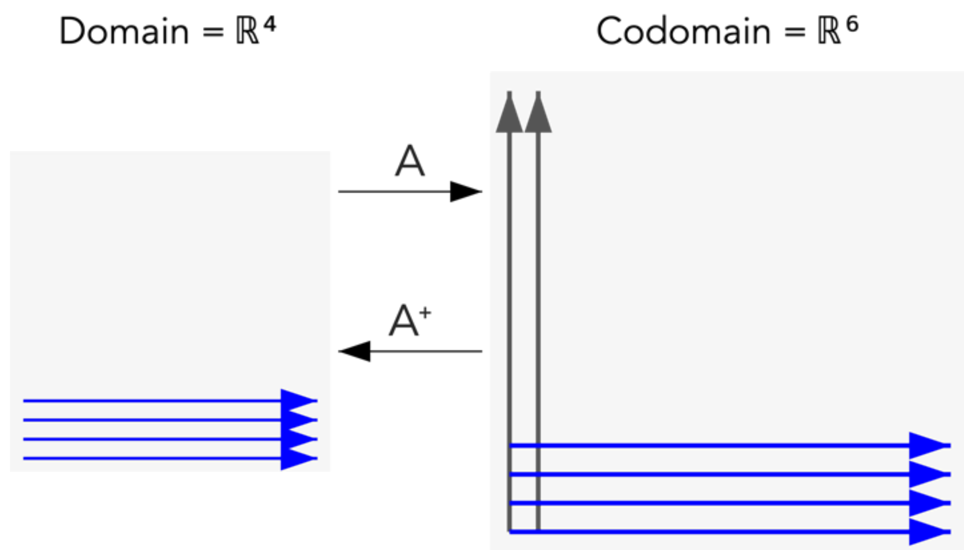
$$(A^T A)^{-1} (A^T A) \vec{x} = \vec{x}$$

② maps vectors from ℓ-null space(A) to $\vec{0}$

suppose $A^T \vec{z} = \vec{0}$, meaning $\vec{z} \in$ ℓ-null space(A)

$$A^+ \vec{z} = (A^T A)^{-1} A^T \vec{z} = (A^T A)^{-1} \vec{0} = \vec{0}$$

So A^+ reverses A on the reachable part of the codomain and sends the unreachable part to $\vec{0}$



Because $A^+ A = I$, this type of pseudoinverse is called a left pseudoinverse

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Wide full-row-rank A

A is a wide full-row-rank (m=4)×(n=6) matrix

• dim(row space) = m
• dim(null space) = n − m
↓
some domain vectors are mapped to $\vec{0}$
• dim(col space) = m
• dim(ℓ-null space) = 0
↓
col space(A) fills codomain

• every codomain vector is produced by at least one domain vector
• some codomain vectors come from many domain vectors
• A is onto but not one-to-one
↓
there is no inverse matrix that maps codomain vectors back to unique domain vectors

However, part of the transformation can be reversed
• even though vectors in null space(A) are sent to $\vec{0}$ by A
• every codomain vector comes from a unique vector in row space(A)

So we can try to reverse A back only into row space(A)

For this type of matrix,
$A^+ = A^T (A A^T)^{-1}$
is called the pseudoinverse of A

(note that $(A A^T)^{-1}$ is defined because A has full row rank)

A^+ acts as follows:
① maps vectors from the codomain back to the corresponding vectors in row space(A):
suppose $\vec{y}$ is any codomain vector
$\vec{x} = A^+ \vec{y}$ belongs to row space(A)
and $A \vec{x} = A A^+ \vec{y} = A A^T (A A^T)^{-1} \vec{y} = \vec{y}$

② ignores the null-space part of the domain

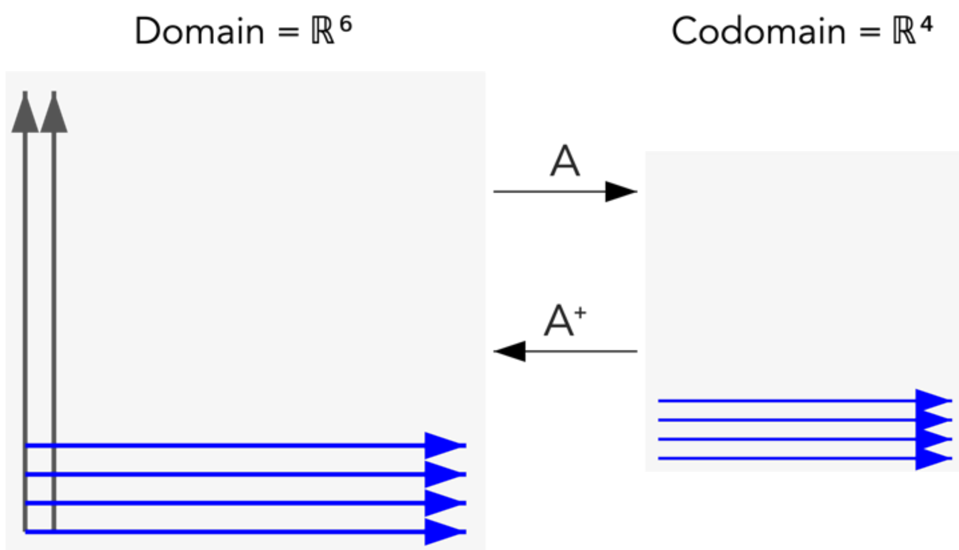
suppose $\vec{x} = \vec{x}_r + \vec{z}$ where $\vec{x}_r \in$ row space(A) and $\vec{z} \in$ null space(A)

then $A\,\vec{x} = A\,\vec{x}_r + A\,\vec{z} = A\,\vec{x}_r + \vec{0} = A\,\vec{x}_r$

so A cannot distinguish $\vec{x}$ from $\vec{x}_r$

A^+ returns the row-space vector $\vec{x}_r$

So A^+ reverses A back to row space(A)
and ignores the null-space directions that A destroys



Because $A\,A^+ = I$, this type of pseudoinverse is called a right pseudoinverse

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General case: neither full column rank nor full row rank

A is a rank-deficient (m=6)×(n=7) matrix with rank r=4

• dim(row space) = r

• dim(null space) = n − r

↓

some domain vectors are mapped to $\vec{0}$

• dim(col space) = r

• dim(ℓ-null space) = m − r

↓

col space(A) does not fill codomain

• some domain vectors are destroyed by A

• some codomain vectors are unreachable

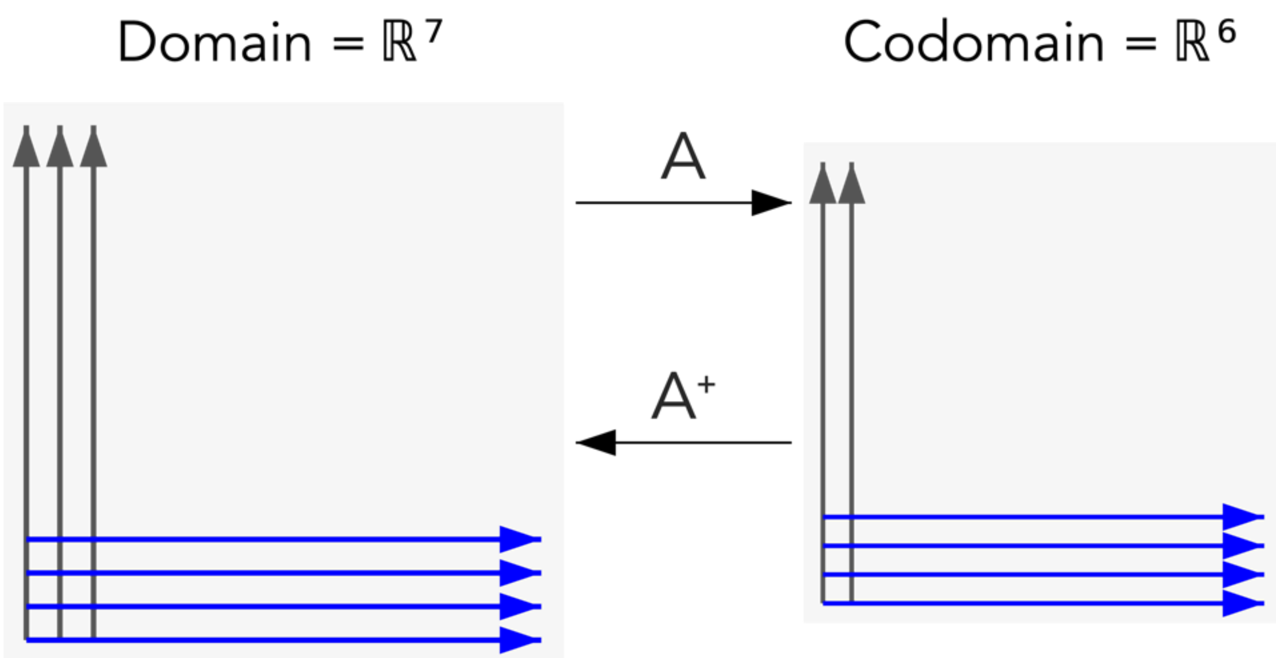
• A is neither one-to-one nor onto

↓

there is no inverse matrix that maps all codomain vectors back to domain vectors

Still, part of the transformation can be reversed

Computation of the pseudoinverse will be introduced later



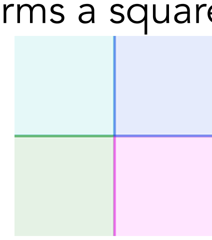
A & A^+ as a bridge between $\mathbb{R}^2$ & $\mathbb{R}^3$

Below is a less abstract example of pseudoinverse A^+

$$3\times 2 \text{ rank-2 matrix } A = [\, \vec{a}_1 \mid \vec{a}_2 \,] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

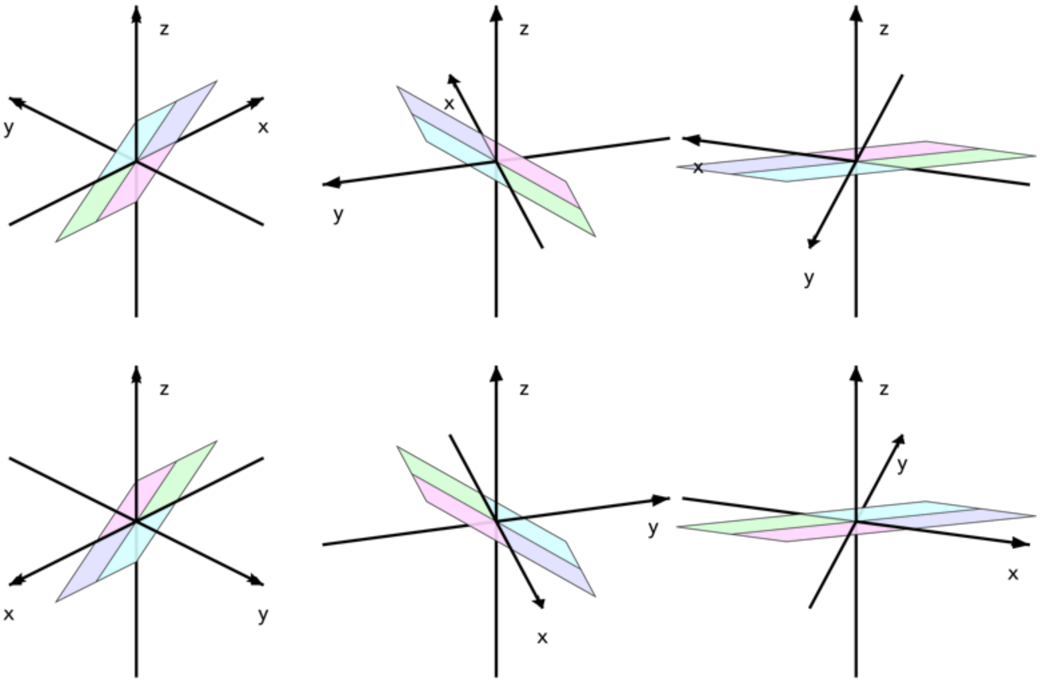
- Domain(A) = $\mathbb{R}^2$
- Codomain(A) = $\mathbb{R}^3$
- col space(A) is a plane in $\mathbb{R}^3$
- ℓ-null space(A) is the direction orthogonal to that plane

A transforms a square unit grid



as follows:

- deforms the square into a parallelogram
- embeds the parallelogram as a plane in $\mathbb{R}^3$



$$A^+ = (A^T A)^{-1} A^T = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^+ A = I_2$$

↓

- A^+ maps every vector in col space(A) back to its original coordinates in $\mathbb{R}^2$
- A^+ returns the unique vector $\vec{x}$ in row space(A) that produced $A\vec{x} = \vec{y}$
- A^+ also maps every vector orthogonal to the plane to $\vec{0}$