Algebraic & geometric definition

M^{-1} exists for any full-rank square M and
is defined as $M^{-1} \times M = M \times M^{-1} = I$ (identity matrix)

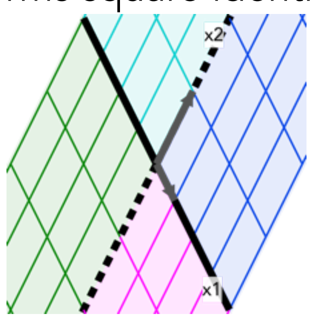
$\Leftrightarrow$

M^{-1} reverses transformation by M

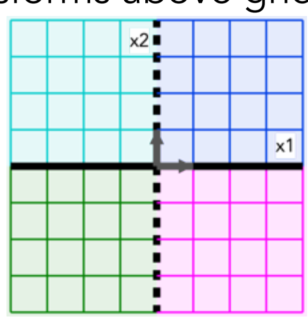
$$\text{Consider example of full-rank } M = \begin{bmatrix} 0.5 & 1 \\ -1 & 2 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 1 & -0.5 \\ 0.5 & 0.25 \end{bmatrix}$$

M transforms square identity grid to



M^{-1} transforms above grid back to



Computing M^{-1} by row reduction of [M | I]

Suppose
- M is a full-rank square matrix
- I is identity matrix of same size

Will compose
- concatenated matrix [M | I]

$$\left[\begin{array}{ccc|ccc} m_{11} & m_{12} & m_{13} & 1 & 0 & 0 \\ m_{21} & m_{22} & m_{23} & 0 & 1 & 0 \\ m_{31} & m_{32} & m_{33} & 0 & 0 & 1 \end{array}\right]$$

- denote its reduced echelon form as [N | J]

Suppose row reduction algorithm takes k steps and
each step is equivalent to left-multiplication by elementary matrix E

$\Downarrow$

[N | J] = (Ek…E1) [M | I] =

[(Ek…E1) M | (Ek…E1) I]

From the definition of reduced row echelon form follows that
if [N | J] is in this form, so is N

N = (Ek…E1) M

$\Downarrow$

(Ek...E1) are the matrices to transform M to reduced echelon

Recall that if M is full-rank, its reduced echelon equivalent is I

$\Downarrow$

N = I

also note that J = (Ek...E1)

Product JM = (Ek...E1) M = I

$\Downarrow$

J = left-inverse of M

Next, will show that J is right-inverse of M:

from (Ek...E1) M = I

follows

M = $(Ek...E1)^{-1}$

(structure of elementary matrices described in row reduction tutorial shows that they are full-rank and invertible, therefore (Ek...E1) is invertible)

$\Downarrow$

MJ = $(Ek...E1)^{-1}$(Ek...E1) = I

Uniqueness of M^{-1}

Suppose full-rank square matrix M has two inverses, A & B so that

MA = I & MB = I

MA - MB = M(A - B) = [0]

or every column of A - B is in the null space of M

since M is full-rank, its null-space is {0}, therefore A - B = 0

M^{-1} is defined only for square full-rank M

① Size requirement:
To satisfy definition $MM^{-1} = I$ & $M^{-1}M = I$,
M has to be square

② Rank requirement: suppose
• non-full-rank matrix M = [m1 | m2 | c1m1+c2m2]
• M has a left-sided inverse A, thus, AM = I
or
A[m1 | m2 | c1m1+c2m2] = I =
[Am1 | Am2 | c1Am1 + c2Am2] = I

Since linear dependence is preserved under left-multiplication,
AM has linearly dependent columns, and cannot equal identity
⇓
non-full-rank matrix cannot have a left inverse

Same rank-2 matrix M can be presented as

$$\begin{bmatrix} m1 \\ m2 \\ c1m1+c2m2 \end{bmatrix}$$

• Suppose M has a right-sided inverse A, thus MA = I

By similar logic, MA will have linearly dependent rows and cannot equal identity

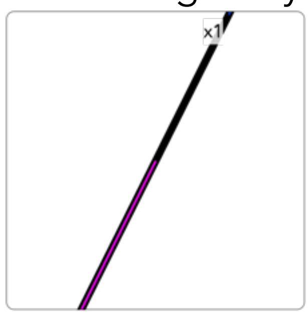
$\Downarrow$

non-full-rank matrix cannot have a right inverse

Geometric intuition:

Consider example of rank-1 matrix M = $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Transformation of 2D grid by M is shown:



Even though M represents transfirmation from $\mathbb{R}^2$ to $\mathbb{R}^2$,

‣ all original vectors are mapped on a single line (column space of M)

‣ all information in the direction orthogonal to that line (null space of M) is lost

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Algebraic properties of M^{-1}

① Let A and B be invertible square matrices of the same size

• Claim: $(AB)^{-1} = B^{-1}A^{-1}$

• Proof: Consider the product $(AB)(B^{-1}A^{-1})$

Using associativity, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

Since $BB^{-1} = I$, this simplifies to $AIA^{-1} = AA^{-1} = I$
Similarly, $(B^{-1}A^{-1})(AB) = I$
Therefore, $B^{-1}A^{-1}$ is the inverse of AB

② Let A be an invertible square matrix
- Claim: $(A^T)^{-1} = (A^{-1})^T$
- Proof: $(A^T)^{-1}A^T = (AA^{-1})^T = I^T = I$

Thus, $(A^{-1})^T$ is a left inverse of A^T
A similar argument shows it is also a right inverse

③ Let A be an invertible square matrix
- Claim: $(A^{-1})^{-1} = A$
- Proof: By definition, $A \times A^{-1} = I$

So A is the inverse of A^{-1}
By uniqueness of the inverse, $(A^{-1})^{-1} = A$

④ Let A be an invertible square matrix and let λ be a nonzero scalar
- Claim: $(\lambda A)^{-1} = \left(\frac{1}{\lambda}\right) A^{-1}$
- Proof: Multiply (λA) by $\left(\frac{1}{\lambda}\right) A^{-1}$

$$\lambda A \left(\frac{1}{\lambda}\right) A^{-1} = \lambda \times \left(\frac{1}{\lambda}\right) (AA^{-1}) = 1 \times I = I$$

Similarly, $\left(\frac{1}{\lambda}\right) A^{-1} \lambda A = I$

Therefore, $(\lambda A)^{-1} = \left(\frac{1}{\lambda}\right) A^{-1}$

⑤ Let A be an invertible square matrix

- Claim: $I^{-1} = I$
- Proof: $I \times I = I$

So I is its own inverse

Visual example of $(AB)^{-1}$

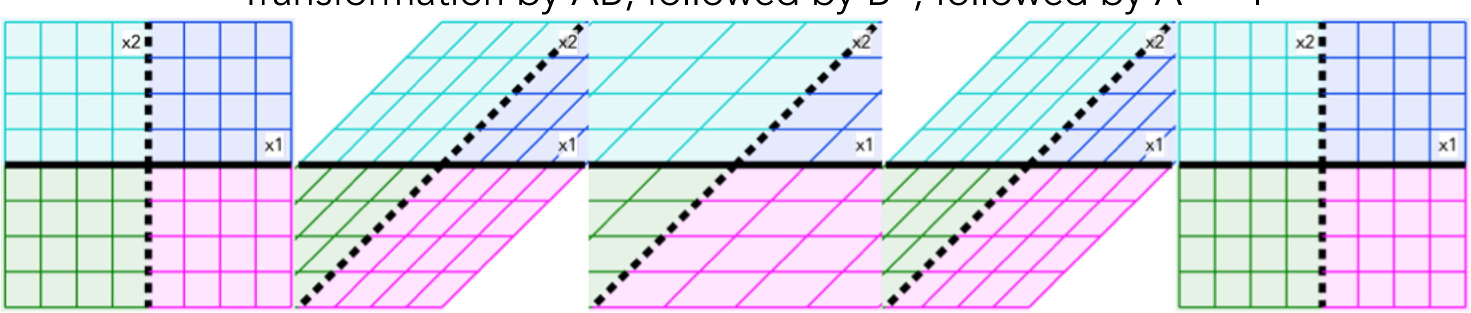
Consider example:

Matrix A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (shear)

Matrix B = $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (scaling)

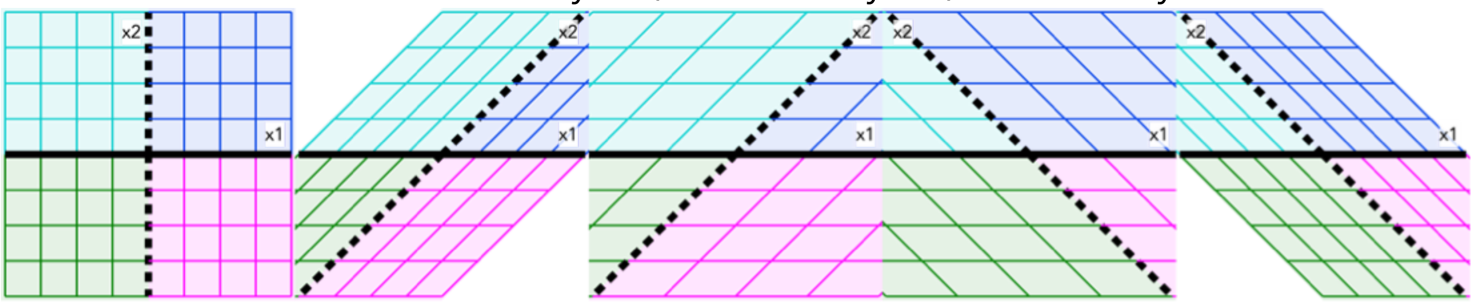
Images show correct sequence:

- Identity grid
- Transformation by A
- Transformation by AB
- Transformation by AB, followed by B^{-1} = A
- Transformation by AB, followed by B^{-1}, followed by A^{-1} = I



Images show reverse and incorrect sequence:

- Identity grid
- Transformation by A
- Transformation by AB
- Transformation by AB, followed by A^{-1}
- Transformation by AB, followed by A^{-1}, followed by B^{-1}



M^{-1} as solution to $A \times \vec{x} = \vec{b}$

① Let A be an n×n matrix, $\vec{x}$ be a vector in $\mathbb{R}^n$, and $\vec{b}$ be a vector in $\mathbb{R}^n$

Consider the linear system $A \times \vec{x} = \vec{b}$

② Assume A is invertible

- Claim: The system $A \times \vec{x} = \vec{b}$ has exactly one solution, namely $\vec{x} = A^{-1}\vec{b}$
- Proof: Start from $A \times \vec{x} = \vec{b}$

Left-multiply both sides by A^{-1}

$A^{-1}(A \times \vec{x}) = A^{-1}\vec{b}$

Using associativity, $(A^{-1}A) \times \vec{x} = A^{-1}\vec{b}$

Since $A^{-1}A = I$, we get $I \times \vec{x} = A^{-1}\vec{b}$

Since $I \times \vec{x} = \vec{x}$, we conclude $\vec{x} = A^{-1}\vec{b}$

③ Uniqueness of the solution

• Claim: If A is invertible, $A\times\vec{x} = \vec{b}$ cannot have two different solutions

• Proof: Suppose $A\times\vec{x}_1 = \vec{b}$ and $A\times\vec{x}_2 = \vec{b}$

Subtract the equations: $A\times\vec{x}_1 - A\times\vec{x}_2 = \vec{b} - \vec{b}$

Factor: $A\times(\vec{x}_1 - \vec{x}_2) = \vec{0}$

Left-multiply by A^{-1}: $A^{-1}\left(A\times(\vec{x}_1 - \vec{x}_2)\right) = A^{-1}\vec{0}$

$(A^{-1}A)\times(\vec{x}_1 - \vec{x}_2) = \vec{0}$

$I\times(\vec{x}_1 - \vec{x}_2) = \vec{0}$

So $\vec{x}_1 - \vec{x}_2 = \vec{0}$, hence $\vec{x}_1 = \vec{x}_2$

④ Interpretation

• Claim: A^{-1} undoes the action of A

• Proof: If $A\times\vec{x} = \vec{b}$, then $\vec{x}$ is the input that produces $\vec{b}$ under the transformation A

Applying A^{-1} reverses this: $\vec{x} = A^{-1}\vec{b}$

⑤ Practical note

$A\times\vec{x} = \vec{b}$ is usually solved with row reducing $[\, A \mid \vec{b}\,]$,

instead of the larger matrix $[\, A \mid I\,]$

⑥ Conclusion

• Row reduction applied to A computes the solution map without reference to any particular $\vec{b}$

• When this map is defined for all $\vec{b}$, it is the inverse

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Trivial cases: 1×1 and 2×2

For $M = \begin{bmatrix} a \end{bmatrix}$ $M^{-1} = \begin{bmatrix} \dfrac{1}{a} \end{bmatrix}$

$$\text{For } M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Above formula

- can be derived by applying row reduction algorithm
- is defined iff $ad - bc \neq 0$