Givens rotation matrix

An m×m rotation matrix G is composed by modifying rows and columns k and i of the identity matrix as follows:

$$G(m, k, i, u, t) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots & \iddots & \vdots \\ 0 & \cdots & c & \cdots & 0 & \cdots & s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & \iddots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \iddots & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -s & \cdots & 0 & \cdots & c & \cdots & 0 \\ \vdots & \iddots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

Will use $\mathbb{R}^5$ as an example

$$\text{Matrix G with modified coordinates 2 \& 4} = \left[\begin{array}{c|c|c|c|c} \vec{e}_1 & \vec{g}_2 & \vec{e}_3 & \vec{g}_4 & \vec{e}_5 \end{array}\right]$$

and a size-compatible vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$

$\mathbb{R}^5$ can be decomposed into 2 orthogonal complementary subspaces:

- $\mathcal{P}$ = span $\{ \vec{e}_2, \vec{e}_4 \}$, denoted as $\mathcal{P}_{24}$
- orthogonal complement $\mathcal{P}\perp$

↓

$\vec{v}$ can be decomposed into

- $\vec{v}\| \in \mathcal{P}_{24}$ or $\vec{v}\| = \begin{bmatrix} 0 \\ v_2 \\ 0 \\ v_4 \\ 0 \end{bmatrix}$
- $\vec{v}\perp \in \mathcal{P}\perp$ or $\vec{v}\perp = \begin{bmatrix} v_1 \\ 0 \\ v_3 \\ 0 \\ v_5 \end{bmatrix}$

$$G\,\vec{v}\perp = \left[\begin{array}{c|c|c|c|c} \vec{e}_1 & \vec{g}_2 & \vec{e}_3 & \vec{g}_4 & \vec{e}_5 \end{array}\right] \begin{bmatrix} v_1 \\ 0 \\ v_3 \\ 0 \\ v_5 \end{bmatrix} = \left[\, \vec{e}_1\, v_1 + \vec{e}_3\, v_3 + \vec{e}_5\, v_5 \,\right] = \begin{bmatrix} v_1 \\ 0 \\ v_3 \\ 0 \\ v_5 \end{bmatrix}$$

$$G\,\vec{v}\| = \left[\, \vec{e}_1 \;\middle|\; \vec{g}_2 \;\middle|\; \vec{e}_3 \;\middle|\; \vec{g}_4 \;\middle|\; \vec{e}_5 \,\right] \begin{bmatrix} 0 \\ v_2 \\ 0 \\ v_4 \\ 0 \end{bmatrix} = \left[\, \vec{g}_2\, v_2 + \vec{g}_4\, v_4 \,\right]$$

- G rotates vectors inside $\mathcal{P}_{24}$
- G leaves vectors in $\mathcal{P}\perp$ unchanged

Algorithm description

Givens QR factorization algorithm employs repeated left-multiplication by specific rotation matrices to eliminate below-diagonal entries, which

- transforms an m×n matrix A to upper-triangular matrix R
- accumulates rotation product Gcum, so R = Gcum A
- Q = Gcum^{-1} = Gcum^{T}

For any below-diagonal entry t = A_{ik}, where i > k

① pivot entry is chosen as the diagonal entry A_{kk} and denoted as 'u'

② following quantities are computed:

- $\rho = \sqrt{u^2 + t^2}$

- $c = \dfrac{u}{\rho}$ (cosine of rotation angle θ)

- $s = \dfrac{t}{\rho}$ (sine of rotation angle θ)

so that $c^2 + s^2 = 1$

③ an m×m rotation matrix G is composed by modifying rows and columns k and i of the identity matrix as follows:

$$G(m, k, i, u, t) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots & \iddots & \vdots \\ 0 & \cdots & c & \cdots & 0 & \cdots & s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & \iddots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \iddots & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -s & \cdots & 0 & \cdots & c & \cdots & 0 \\ \vdots & \iddots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

④ product G A is computed
for clarity, G and A are presented as 5×5 matrices:

$$G\,A = \begin{bmatrix} (\vec{e}_1)^\top \\ (\vec{e}_2)^\top \\ (\vec{g}_3)^\top \\ (\vec{e}_4)^\top \\ (\vec{g}_5)^\top \end{bmatrix} \left[\begin{array}{c|c|c|c|c} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{array}\right] =$$

$$\left[\begin{array}{c|c|c|c|c}
(\vec{e}_1)^\top\ \vec{a}_1 & (\vec{e}_1)^\top\ \vec{a}_2 & (\vec{e}_1)^\top\ \vec{a}_3 & (\vec{e}_1)^\top\ \vec{a}_4 & (\vec{e}_1)^\top\ \vec{a}_5 \\ \hline
(\vec{e}_2)^\top\ \vec{a}_1 & (\vec{e}_2)^\top\ \vec{a}_2 & (\vec{e}_2)^\top\ \vec{a}_3 & (\vec{e}_2)^\top\ \vec{a}_4 & (\vec{e}_2)^\top\ \vec{a}_5 \\ \hline
(\vec{g}_3)^\top\ \vec{a}_1 & (\vec{g}_3)^\top\ \vec{a}_2 & (\vec{g}_3)^\top\ \vec{a}_3 & (\vec{g}_3)^\top\ \vec{a}_4 & (\vec{g}_3)^\top\ \vec{a}_5 \\ \hline
(\vec{e}_4)^\top\ \vec{a}_1 & (\vec{e}_4)^\top\ \vec{a}_2 & (\vec{e}_4)^\top\ \vec{a}_3 & (\vec{e}_4)^\top\ \vec{a}_4 & (\vec{e}_4)^\top\ \vec{a}_5 \\ \hline
(\vec{g}_5)^\top\ \vec{a}_1 & (\vec{g}_5)^\top\ \vec{a}_2 & (\vec{g}_5)^\top\ \vec{a}_3 & (\vec{g}_5)^\top\ \vec{a}_4 & (\vec{g}_5)^\top\ \vec{a}_5
\end{array}\right] =$$

$$\left[\begin{array}{c|c|c|c|c}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \hline
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hline
(\vec{g}_3)^\top\ \vec{a}_1 & (\vec{g}_3)^\top\ \vec{a}_2 & (\vec{g}_3)^\top\ \vec{a}_3 & (\vec{g}_3)^\top\ \vec{a}_4 & (\vec{g}_3)^\top\ \vec{a}_5 \\ \hline
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ \hline
(\vec{g}_5)^\top\ \vec{a}_1 & (\vec{g}_5)^\top\ \vec{a}_2 & (\vec{g}_5)^\top\ \vec{a}_3 & (\vec{g}_5)^\top\ \vec{a}_4 & (\vec{g}_5)^\top\ \vec{a}_5
\end{array}\right]$$

only rows $\vec{g}_k^\top$ and $\vec{g}_i^\top$ of G differ from identity rows

↓

- only rows k and i of A are modified
- all remaining rows are copied unchanged from A

⑤ target entry in the product is eliminated

suppose target entry $t = A_{53}$ is eliminated using pivot entry $u = A_{33}$
therefore rows 3 and 5 of A are modified as follows:

$$\text{new row}_3 = c\cdot\text{row}_3(A) + s\cdot\text{row}_5(A)$$
$$\text{new row}_5 = -s\cdot\text{row}_3(A) + c\cdot\text{row}_5(A)$$

in column 3 this produces:

$$\text{new } A_{53} = -s\cdot u + c\cdot t$$

using $c = \dfrac{u}{\rho}$ and $s = \dfrac{t}{\rho}$ gives:

$$\text{new } A_{53} = -\left(\frac{t}{\rho}\right)u + \left(\frac{u}{\rho}\right)t = 0$$

⑥ the previously obtained zeros in the product are preserved

Example below shows

- intermediate 5×5 matrix $A(\ell)$
- current rotation plane $\mathcal{P}_{53} = \text{span}\{\vec{e}_3, \vec{e}_5\}$
- corresponding matrix $G(\ell)$ to eliminate t

$$A(\ell+1) = G(\ell)\,A(\ell) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & s \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -s & 0 & c \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & u & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & t & a_{54} & a_{55} \end{bmatrix}$$

ⓐ multiplication by $G(\ell)$ changes only pivot row k and target row i

↓

previously created zeros in the same column are preserved

ⓑ for any preceding column $\vec{a}_h$, where $h < k$, entries a_{kh} and a_{ih} are already zero

↓

$\vec{a}_h$ has no component in the current rotation plane $\mathcal{P}_{ki}$

↓

$\vec{a}_h \in \mathcal{P}_{ki}\perp$, so $G(\ell)\vec{a}_h = \vec{a}_h$

↓

previously created zeros in the preceding columns are preserved

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Elimination order

① Elimination of columns proceeds left to right
(reason is outlined on previous page)

② Elimination within a given column does not have a defined order

Suppose we are eliminating column 3 in the following matrix:

$$A(j) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & a_{53} & a_{54} & a_{55} \end{bmatrix} \quad \text{(where j is step index)}$$

To finish column 3, the algorithm must eliminate a_{43} and a_{53}

Before these eliminations, the active part of column 3 is

$$[\ u,\ x,\ t\ ]^T$$

where

- $u = a_{33}$
- $x = a_{43}$
- $t = a_{53}$

Each Givens rotation replaces two entries by

$$[\ \text{pivot},\ \text{target}\ ]^T \rightarrow \left[\ \sqrt{\text{pivot}^2 + \text{target}^2}\ ,\ 0\ \right]^T$$

After both a_{43} and a_{53} are eliminated, the diagonal entry satisfies

$$\rho = \pm\sqrt{u^2 + x^2 + t^2}$$

This expression is symmetric in x and t

↓

either elimination order gives the same diagonal magnitude

Alternative explanation:
- any orthogonal matrix G preserves lengths
- each elimination step affects only the pivot row and target row

↓

the length of the eliminated target entry is absorbed into the pivot entry

Suppose A(j+2) = G(j+1) G(j) A
- G(j) eliminates entry a_{43}
- G(j+1) eliminates entry a_{53}
- sign of ρ is fixed

$$G(j+1)\,G(j)\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \\ a_{53} \end{bmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \\ \rho \\ 0 \\ 0 \end{bmatrix}$$

- entries above the pivot row are unchanged
- entries below the pivot row are eliminated
- diagonal magnitude is fixed by length preservation
- as shown on the previous page,
completed columns to the left are not changed by later steps

↓

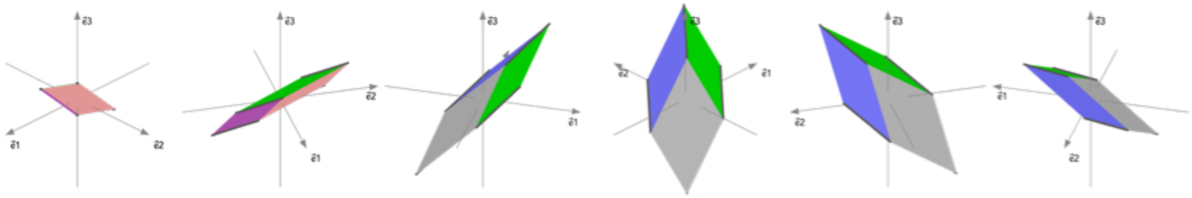
each completed column of R is unique up to sign,
independent of elimination order within that column

Elimination order within column: geometric comparison

$$\text{Two possible orders of eliminating col 1 of A} = \begin{bmatrix} 0.82 & 0.3 & 1.38 \\ 0.92 & 1.48 & 0.2 \\ 0.72 & 0.88 & 1.33 \end{bmatrix}$$

① Bottom to top elimination order

$$\begin{bmatrix} 0.82 & 0.3 & 1.38 \\ 0.92 & 1.48 & 0.2 \\ 0.72 & 0.88 & 1.33 \end{bmatrix}$$

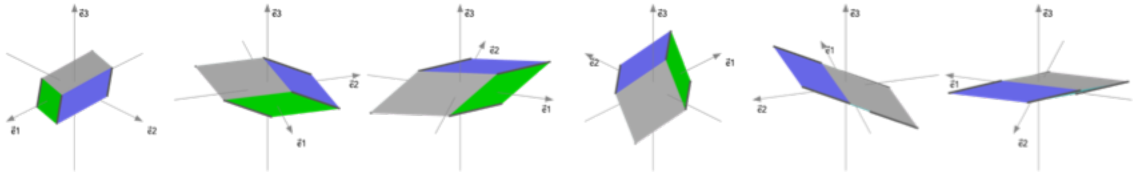


Transformation by G1

- Rotation in the $\vec{e}_1$ & $\vec{e}_3$ plane by $\theta \approx 0.721$ rad
- In $\mathbb{R}^3$, this is equivalent to rotation around $\vec{e}_2$
- $\vec{a}_1$ loses its $\vec{e}_3$ component and lands in the $\vec{e}_1$ & $\vec{e}_2$ coordinate plane

(gray lines track $\vec{a}_1$)

$$\begin{bmatrix} \approx 1.091 & \approx 0.806 & \approx 1.915 \\ 0.92 & 1.48 & 0.2 \\ 0 & \approx 0.463 & \approx 0.089 \end{bmatrix}$$

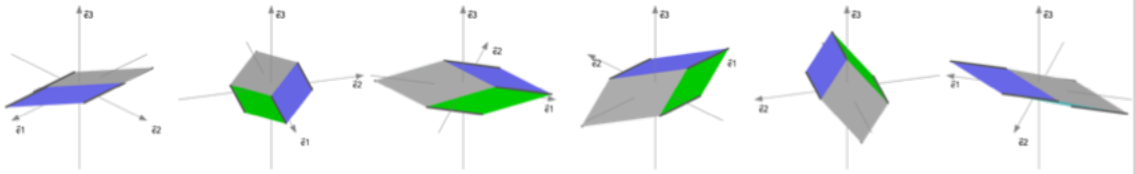


Transformation by G2

- Rotation in the $\vec{e}_1$ & $\vec{e}_2$ plane by $\theta \approx 0.7$ rad
- In $\mathbb{R}^3$, this is equivalent to rotation around $\vec{e}_3$
- $\vec{a}_1$ loses its $\vec{e}_2$ component and is now aligned with the $\vec{e}_1$ line

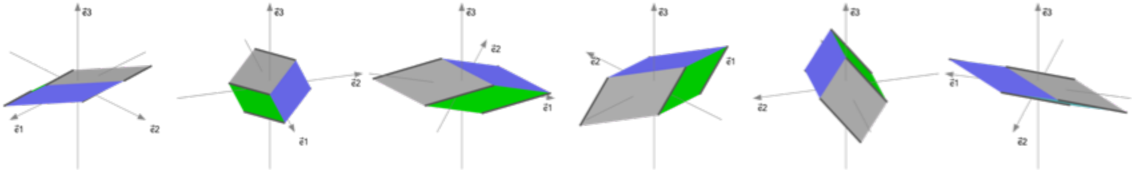
(gray lines track $\vec{a}_1$)

$$\begin{bmatrix} \approx 1.427 & \approx 1.57 & \approx 1.593 \\ 0 & \approx 0.612 & \approx -1.081 \\ 0 & \approx 0.463 & \approx 0.089 \end{bmatrix}$$



The next image shows the same transformed matrix again, but with column 2 highlighted for the next elimination step

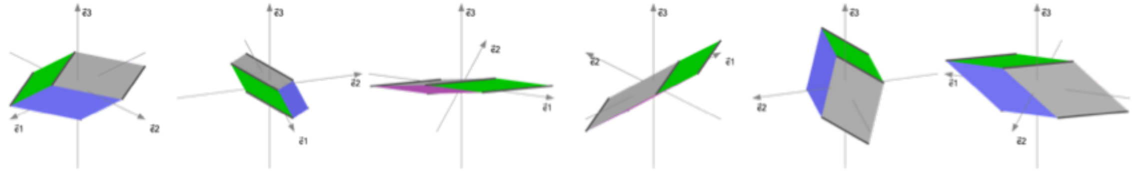
$$\begin{bmatrix} \approx 1.427 & \approx 1.57 & \approx 1.593 \\ 0 & \approx 0.612 & \approx -1.081 \\ 0 & \approx 0.463 & \approx 0.089 \end{bmatrix}$$



Transformation by G3
- Rotation in the $\vec{e}_2$ & $\vec{e}_3$ plane by θ ≈0.648 rad
- In $\mathbb{R}^3$, this is equivalent to rotation around $\vec{e}_1$
- $\vec{a}_2$ loses its $\vec{e}_3$ component and lands in the $\vec{e}_1$ & $\vec{e}_2$ coordinate plane

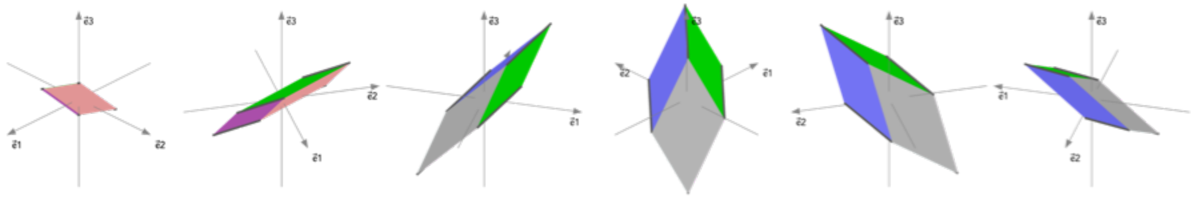
(gray lines track $\vec{a}_2$)

$$\begin{bmatrix} \approx 1.427 & \approx 1.57 & \approx 1.593 \\ 0 & \approx 0.768 & \approx -0.808 \\ 0 & 0 & \approx 0.723 \end{bmatrix}$$



② Top to bottom elimination order

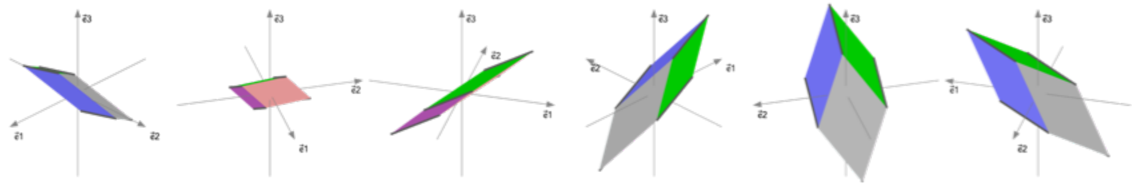
$$\begin{bmatrix} 0.82 & 0.3 & 1.38 \\ 0.92 & 1.48 & 0.2 \\ 0.72 & 0.88 & 1.33 \end{bmatrix}$$



Transformation by G1
- Rotation in the $\vec{e}_1$ & $\vec{e}_2$ plane by θ ≈0.843 rad
- In $\mathbb{R}^3$, this is equivalent to rotation around $\vec{e}_3$
- $\vec{a}_1$ loses its $\vec{e}_2$ component and is now aligned with the $\vec{e}_1$ line

(gray lines track $\vec{a}_1$)

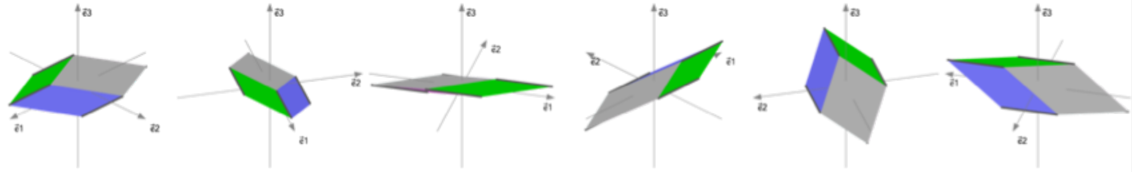
$$\begin{bmatrix} \approx 1.232 & \approx 1.304 & \approx 1.068 \\ 0 & \approx 0.761 & \approx -0.897 \\ 0.72 & 0.88 & 1.33 \end{bmatrix}$$



Transformation by G2
- Rotation in the $\vec{e}_1$ & $\vec{e}_3$ plane by θ ≈0.529 rad
- In $\mathbb{R}^3$, this is equivalent to rotation around $\vec{e}_2$
- $\vec{a}_1$ loses its $\vec{e}_3$ component and lands in the $\vec{e}_1$ & $\vec{e}_2$ coordinate plane

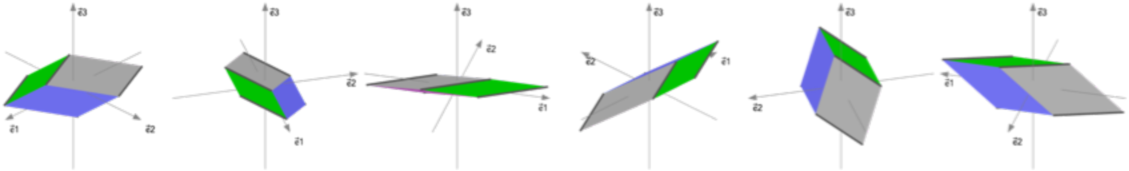
(gray lines track $\vec{a}_1$)

$$\begin{bmatrix} \approx 1.427 & \approx 1.57 & \approx 1.593 \\ 0 & \approx 0.761 & \approx -0.897 \\ 0 & \approx 0.102 & \approx 0.61 \end{bmatrix}$$



The next image shows the same transformed matrix again, but with column 2 highlighted for the next elimination step

$$\begin{bmatrix} \approx 1.427 & \approx 1.57 & \approx 1.593 \\ 0 & \approx 0.761 & \approx -0.897 \\ 0 & \approx 0.102 & \approx 0.61 \end{bmatrix}$$

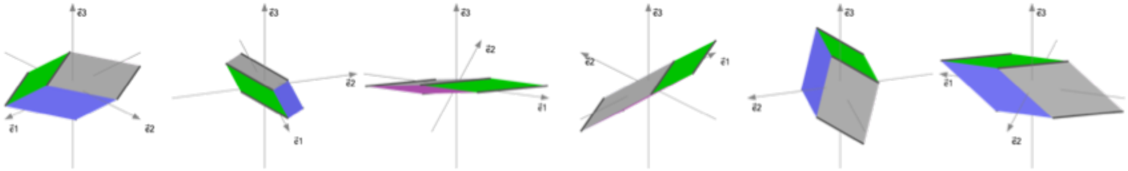


Transformation by G3

- Rotation in the $\vec{e}_2$ & $\vec{e}_3$ plane by $\theta \approx 0.133$ rad
- In $\mathbb{R}^3$, this is equivalent to rotation around $\vec{e}_1$
- $\vec{a}_2$ loses its $\vec{e}_3$ component and lands in the $\vec{e}_1$ & $\vec{e}_2$ coordinate plane

(gray lines track $\vec{a}_2$)

$$\begin{bmatrix} \approx 1.427 & \approx 1.57 & \approx 1.593 \\ 0 & \approx 0.768 & \approx -0.808 \\ 0 & 0 & \approx 0.723 \end{bmatrix}$$



- Compare the bordered images after elimination of the first column where $\vec{a}_1$ aligns with $\vec{e}_1$:

The intermediate transformations are different, showing that column 1 can be aligned with the $\vec{e}_1$ axis in more than one way

- Compare the final matrices.

After the algorithm is completed, the final upper-triangular matrix R is the same with

the only remaining ambiguity is the usual sign choice

- Teaching analogy:

Like a Rubik's cube, different sequences of moves can pass through different intermediate states,

but still arrive at the same final solved state

Comparison with Gram-Schmidt algorithm

	Givens rotation	Gram-Schmidt orthogonalization
Process	• transforms A to R by left-multiplication • Gcum is accumulated from rotations • $Q = (Gcum)^T$	• transforms columns of A into Q • R is computed at the end as Q^TA
Sign of diagonal entries of R	always non-negative	any
Dimensions of Q & R	usually full: Q is m×m, R is m×n	usually thin: Q is m×n, R is n×n
Step operation	rotates two rows	orthogonalizes and normalizes one column
How entries of R appear	below-diagonal entries are eliminated directly	entries of R are projection coefficients of column $\vec{a}_j$ onto preceding $\hat{q}_i$ directions
Geometric idea	rotation in a coordinate plane	• subtracting projection onto previous directions • normalization
Numerical stability	stable	can lose orthogonality if • columns are nearly dependent • residual vectors become short
Efficiency for dense matrices	similar	similar
Efficiency for sparse matrices	superior due to fewer steps required	inferior

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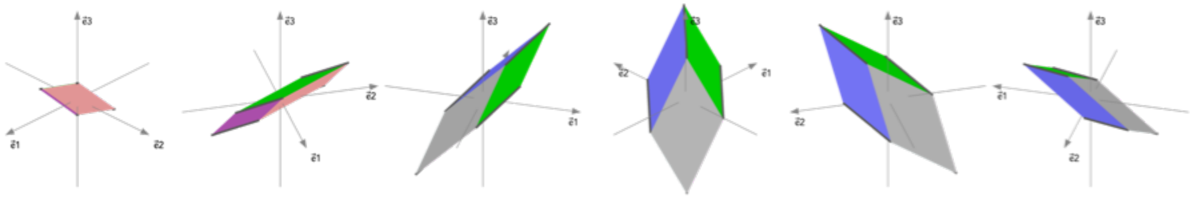
Numerical example

$$A = \begin{bmatrix} 0.82 & 0.3 & 1.38 \\ 0.92 & 1.48 & 0.2 \\ 0.72 & 0.88 & 1.33 \end{bmatrix}$$

$$\begin{bmatrix} 0.82 & 0.3 & 1.38 \\ 0.92 & 1.48 & 0.2 \\ 0.72 & 0.88 & 1.33 \end{bmatrix}$$

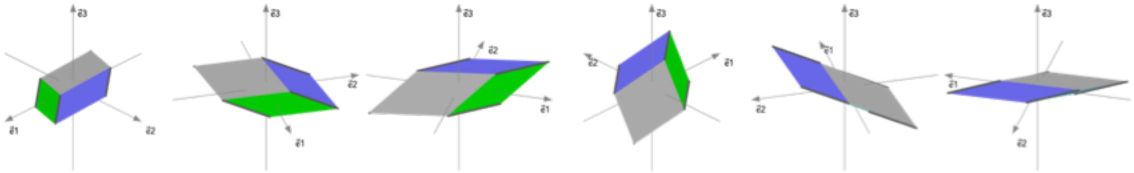


Transformation by G1

- Rotation in the $\vec{e}_1$ & $\vec{e}_3$ plane by θ ≈0.721 rad
- In $\mathbb{R}^3$, this is equivalent to rotation around $\vec{e}_2$
- $\vec{a}_1$ loses its $\vec{e}_3$ component and lands in the $\vec{e}_1$ & $\vec{e}_2$ coordinate plane

(gray lines track $\vec{a}_1$)

$$\begin{bmatrix} \approx 1.091 & \approx 0.806 & \approx 1.915 \\ 0.92 & 1.48 & 0.2 \\ 0 & \approx 0.463 & \approx 0.089 \end{bmatrix}$$

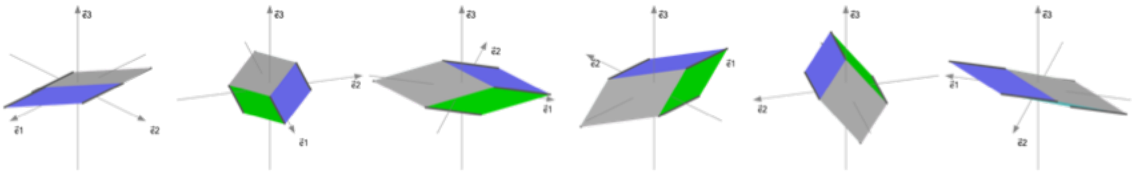


Transformation by G2

- Rotation in the $\vec{e}_1$ & $\vec{e}_2$ plane by θ ≈0.7 rad
- In $\mathbb{R}^3$, this is equivalent to rotation around $\vec{e}_3$
- $\vec{a}_1$ loses its $\vec{e}_2$ component and is now aligned with the $\vec{e}_1$ line

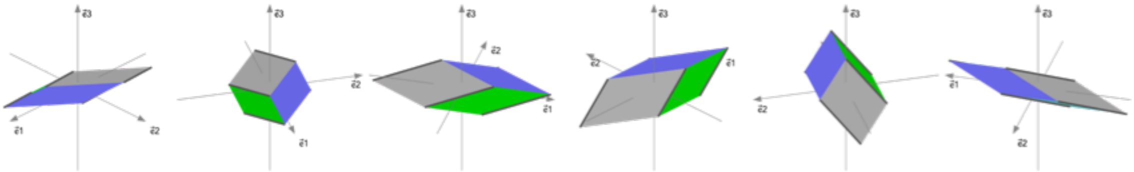
(gray lines track $\vec{a}_1$)

$$\begin{bmatrix} \approx 1.427 & \approx 1.57 & \approx 1.593 \\ 0 & \approx 0.612 & \approx -1.081 \\ 0 & \approx 0.463 & \approx 0.089 \end{bmatrix}$$



The next image shows the same transformed matrix again, but with column 2 highlighted for the next elimination step

$$\begin{bmatrix} \approx 1.427 & \approx 1.57 & \approx 1.593 \\ 0 & \approx 0.612 & \approx -1.081 \\ 0 & \approx 0.463 & \approx 0.089 \end{bmatrix}$$

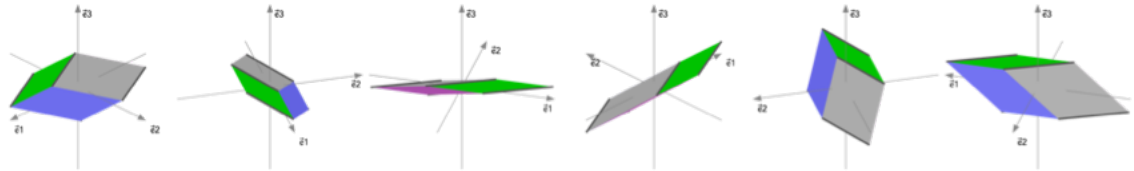


Transformation by G3

- Rotation in the $\vec{e}_2$ & $\vec{e}_3$ plane by θ ≈0.648 rad
- In $\mathbb{R}^3$, this is equivalent to rotation around $\vec{e}_1$
- $\vec{a}_2$ loses its $\vec{e}_3$ component and lands in the $\vec{e}_1$ & $\vec{e}_2$ coordinate plane

(gray lines track $\vec{a}_2$)

$$\begin{bmatrix} \approx 1.427 & \approx 1.57 & \approx 1.593 \\ 0 & \approx 0.768 & \approx -0.808 \\ 0 & 0 & \approx 0.723 \end{bmatrix}$$



$$G1 = \begin{bmatrix} \frac{a_{11}}{\sqrt{a_{11}^2+a_{31}^2}} & 0 & \frac{a_{31}}{\sqrt{a_{11}^2+a_{31}^2}} \\ 0 & 1 & 0 \\ -\frac{a_{31}}{\sqrt{a_{11}^2+a_{31}^2}} & 0 & \frac{a_{11}}{\sqrt{a_{11}^2+a_{31}^2}} \end{bmatrix} = \begin{bmatrix} \approx 0.751 & 0 & \approx 0.66 \\ 0 & 1 & 0 \\ \approx -0.66 & 0 & \approx 0.751 \end{bmatrix}$$

$$G2 = \begin{bmatrix} \frac{a_{11}}{\sqrt{a_{11}^2+a_{21}^2}} & \frac{a_{21}}{\sqrt{a_{11}^2+a_{21}^2}} & 0 \\ -\frac{a_{21}}{\sqrt{a_{11}^2+a_{21}^2}} & \frac{a_{11}}{\sqrt{a_{11}^2+a_{21}^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \approx 0.765 & \approx 0.645 & 0 \\ \approx -0.645 & \approx 0.765 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

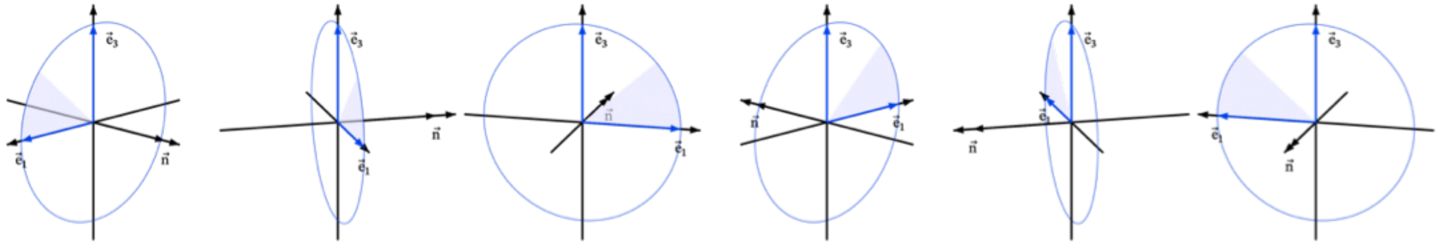
$$G3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{a_{22}}{\sqrt{a_{22}^2+a_{32}^2}} & \frac{a_{32}}{\sqrt{a_{22}^2+a_{32}^2}} \\ 0 & -\frac{a_{32}}{\sqrt{a_{22}^2+a_{32}^2}} & \frac{a_{22}}{\sqrt{a_{22}^2+a_{32}^2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \approx 0.797 & \approx 0.604 \\ 0 & \approx -0.604 & \approx 0.797 \end{bmatrix}$$

Rotation matrices visualized (same numerical example)

$$A = \begin{bmatrix} 0.82 & 0.3 & 1.38 \\ 0.92 & 1.48 & 0.2 \\ 0.72 & 0.88 & 1.33 \end{bmatrix}$$

$$G1 = \begin{bmatrix} \approx 0.751 & 0 & \approx 0.66 \\ 0 & 1 & 0 \\ \approx -0.66 & 0 & \approx 0.751 \end{bmatrix}$$

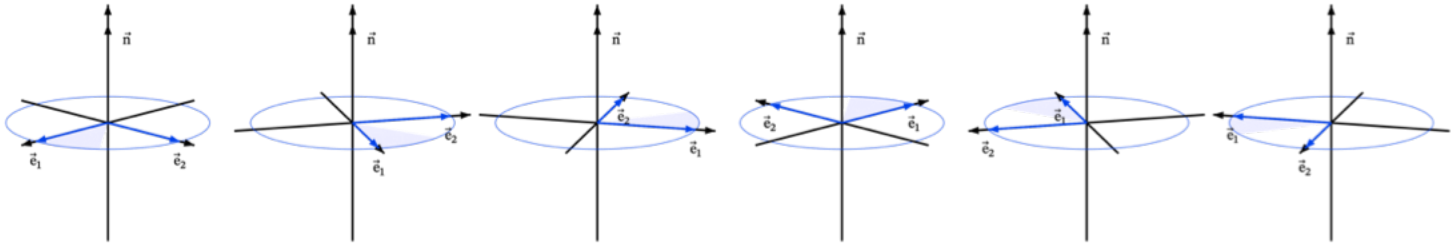
- Rotation in the $\vec{e}_1$ & $\vec{e}_3$ plane by θ ≈0.721 rad
- In $\mathbb{R}^3$, this is equivalent to rotation around $\vec{e}_2$
- $\vec{a}_1$ loses its $\vec{e}_3$ component and lands in the $\vec{e}_1$ & $\vec{e}_2$ coordinate plane



$$G2 = \begin{bmatrix} \approx 0.765 & \approx 0.645 & 0 \\ \approx -0.645 & \approx 0.765 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

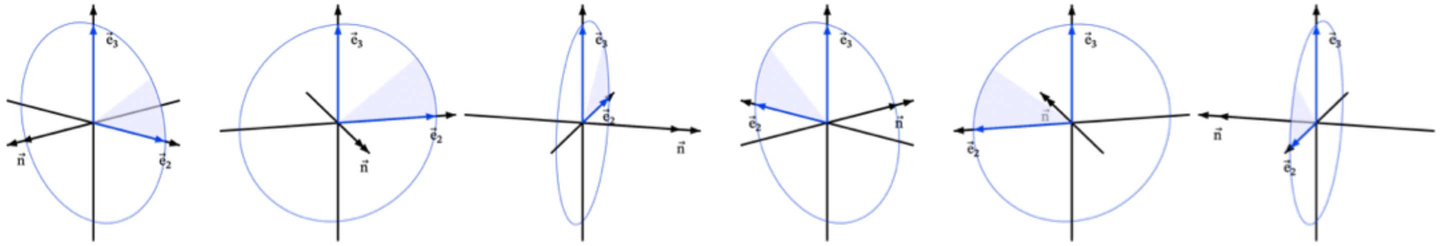
- Rotation in the $\vec{e}_1$ & $\vec{e}_2$ plane by θ ≈0.7 rad
- In $\mathbb{R}^3$, this is equivalent to rotation around $\vec{e}_3$

- $\vec{a}_1$ loses its $\vec{e}_2$ component and is now aligned with the $\vec{e}_1$ line



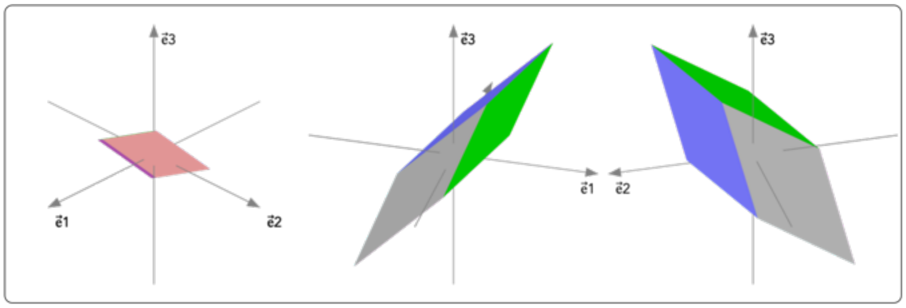
$$G3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \approx 0.797 & \approx 0.604 \\ 0 & \approx -0.604 & \approx 0.797 \end{bmatrix}$$

- Rotation in the $\vec{e}_2$ & $\vec{e}_3$ plane by $\theta \approx 0.648$ rad
- In $\mathbb{R}^3$, this is equivalent to rotation around $\vec{e}_1$
- $\vec{a}_2$ loses its $\vec{e}_3$ component and lands in the $\vec{e}_1$ & $\vec{e}_2$ coordinate plane



Visual comparison between GS orthogonalization & Givens rotation

$$A = \begin{bmatrix} 0.82 & 0.3 & 1.38 \\ 0.92 & 1.48 & 0.2 \\ 0.72 & 0.88 & 1.33 \end{bmatrix} = Q\,R = \begin{bmatrix} \approx 0.575 & \approx -0.784 & \approx -0.234 \\ \approx 0.645 & \approx 0.61 & \approx -0.461 \\ \approx 0.504 & \approx 0.115 & \approx 0.856 \end{bmatrix} \begin{bmatrix} \approx 1.427 & \approx 1.57 & \approx 1.593 \\ 0 & \approx 0.768 & \approx -0.808 \\ 0 & 0 & \approx 0.723 \end{bmatrix}$$



GS orthogonalization:
Serial column shear and scaling
reshape the figure so that

- columns become orthogonal
- each column has unit length

↓

- A is transformed into Q
- R is computed as $Q^T A$

Givens rotation:
Serial row rotations
reorient the figure so that

- $\vec{a}_1$ aligns with $\vec{e}_1$
- $\vec{a}_2$ lies in the $\vec{e}_1 \, \vec{e}_2$ plane

↓

- A is transformed into R
- $Q = (G_j \ldots G_1) - 1$