Matrices with one repeated λ

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}$

Solving $\det(A - \lambda I) = 0$:

$$(a - \lambda)(d - \lambda) - bc = 0$$

$$\downarrow$$

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

$$\downarrow$$

$$\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}$$

$$\downarrow$$

A has a single λ if

$$(a + d)^2 - 4(ad - bc) = 0$$

$$\downarrow$$

$$a^2 - 2ad + d^2 + 4bc = 0$$

$$\downarrow$$

$$\lambda = \frac{a + d}{2}$$

case ①: $b \neq 0$

$$\downarrow$$

$$c = -\frac{(a - d)^2}{4b}$$

$$\downarrow$$

A with a single λ and $b \neq 0$ has the form: $\begin{bmatrix} a & b \\ -\dfrac{(a-d)^2}{4b} & d \end{bmatrix}$

To find the eigenvector(s), we solve the homogeneous system

$$(A - \lambda I)\,\vec{x} = \begin{bmatrix} a - \dfrac{a+d}{2} & b \\ -\dfrac{(a-d)^2}{4b} & d - \dfrac{a+d}{2} \end{bmatrix} \vec{x} = \begin{bmatrix} \dfrac{a-d}{2} & b \\ -\dfrac{(a-d)^2}{4b} & \dfrac{d-a}{2} \end{bmatrix} \vec{x} = \vec{0}$$

λ was computed so that $\det(A - \lambda I) = 0$

↓

$$\text{row}_2(A - \lambda I) = -\left(\frac{a-d}{2b}\right) \text{row}_1(A - \lambda I)$$

↓

we have one system (for the single λ) with one free variable

↓

eigenspace(A) has one dimension

Use the first row and choose free variable $x_1 = 2b$:

$$\left(\frac{a-d}{2}\right) x_1 + b\,x_2 = 0$$

↓

$$(a - d)\,b + b\,x_2 = 0$$

↓

$x_2 = d - a$

$\downarrow$

One eigenvector $\vec{x} = \begin{bmatrix} 2b \\ d - a \end{bmatrix}$

One particular form of such A has a = d, b = 1, c = 0:

so-called Jordan block $J = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ with the eigendirection $\vec{e}_1$

case ②: the transpose of case ①

The corresponding particular form is $J^T = \begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix}$ with the eigendirection $\vec{e}_2$

case ③: b = 0 and c = 0

$\downarrow$

we obtain one uniform scaling matrix $S = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

$a = d$ and $\lambda = a$

$\downarrow$

$$A - \lambda I = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$\text{Solving } (A - \lambda I)\,\vec{x} = \vec{0}: \quad \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

↓

Both equations are always true

↓

x_1 and x_2 are both free variables

↓

every nonzero vector $\vec{x}$ is an eigenvector

True for case ①, ②, and ③:

$(A - \lambda I)^2$ is a zero matrix

$$\text{Let } u = \frac{a - d}{2}$$

The single λ condition

$(a - d)^2 + 4bc = 0$

becomes

$u^2 + bc = 0$

$$\text{Denote } B = A - \lambda I = \left[\begin{array}{c|c} u & b \\ \hline c & -u \end{array}\right]$$

$$B^2 = \begin{bmatrix} u & b \\ c & -u \end{bmatrix} \begin{bmatrix} u & b \\ c & -u \end{bmatrix} = \begin{bmatrix} u^2 + bc & u\,b - b\,u \\ c\,u - u\,c & cb + u^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore

$$(A - \lambda I)^2 = 0$$

If A is not a uniform scaling matrix, rank(B) = 1,
so there is a vector $\vec{w}$ such that B $\vec{w} \neq \vec{0}$

Denote $\vec{v} = B\,\vec{w}$

Since B^2 is a zero matrix, $B\,\vec{v} = B\,B\,\vec{w} = \vec{0}$

$$\downarrow$$

$$(A - \lambda I)\,\vec{v} = \vec{0}$$

$$\downarrow$$

equivalently
every nonzero vector produced by B is an eigenvector of A

Relevance of this concept, as well as J and J^T, will be shown on the next page

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Similarity transformation of Jordan block

Similarity transformation is a transformation of type:

$$A = C\,J\,C^{-1}$$

$$\downarrow$$

$$J = C^{-1}\,A\,C$$

Suppose $J \vec{x} = \lambda \vec{x}$

↓

$C^{-1} A C \vec{x} = \lambda \vec{x}$

↓

$A C \vec{x} = C \lambda \vec{x}$

↓

$A (C \vec{x}) = \lambda (C \vec{x})$

↓

- λ is an eigenvalue of both J and A
- $(C \vec{x})$ is an eigenvector of A

If J^n can be easily computed,

similarity transformation allows us to compute powers of A as

$$A^n = C \; J^n \; C^{-1}$$

On the previous page, we showed that every nonzero vector produced by $B = A - \lambda I$

is an eigenvector of A

Choose $\vec{w}$ so that $B \vec{w} \neq \vec{0}$

Define $\vec{v} = B \vec{w}$

↓

- $B \vec{w} = \vec{v}$
- $A \vec{v} = \lambda \vec{v}$

The vector $\vec{w}$ is called a generalized eigenvector:

$\vec{w}$ is not an eigenvector of A, but $(A - \lambda I) \vec{w}$ is an eigenvector of A

$B \vec{w} \neq \vec{0}$, so $\vec{w}$ cannot be a scalar multiple of $\vec{v}$

↓

$C = [\, \vec{v} \mid \vec{w} \,]$ is invertible

Rearranging the similarity condition:
$A\,C = C\,J$

$A\,C = A\,[\,\vec{v} \mid \vec{w}\,] = [\,A\,\vec{v} \mid A\,\vec{w}\,]$

Since $B\,\vec{w} = \vec{v}$,
$(A - \lambda\,I)\,\vec{w} = \vec{v}$
↓
$A\,\vec{w} = \lambda\,\vec{w} + \vec{v}$
and $A\,\vec{v} = \lambda\,\vec{v}$
↓
$A\,C = [\,\lambda\,\vec{v} \mid \vec{v} + \lambda\,\vec{w}\,]$

Note that $A\,C = [\,\lambda\,\vec{v} \mid \vec{v} + \lambda\,\vec{w}\,] = [\,\vec{v} \mid \vec{w}\,] \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = C \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

↓

$C = [\,\vec{v} \mid \vec{w}\,]$ and $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ satisfy the similarity condition $A\,C = C\,J$

Therefore
$A = C\,J\,C^{-1}$

The scales of $\vec{w}$ and $\vec{v}$ are coordinated by $\vec{v} = B\,\vec{w}$
Once the scale of $\vec{w}$ is chosen, the scale of $\vec{v}$ is determined
This differs from ordinary diagonalization, where the eigenvectors may be scaled independently

For $n \geq 2$,

For $n \geq 2$, $A^n = C\, J^n\, C^{-1}$ and $J^n = \begin{bmatrix} \lambda^n & n\,\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$

More generally, if $\lambda \neq 0$ and λ^t is defined (with a branch chosen when necessary),

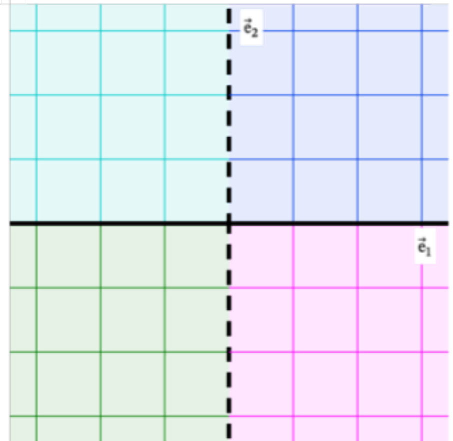
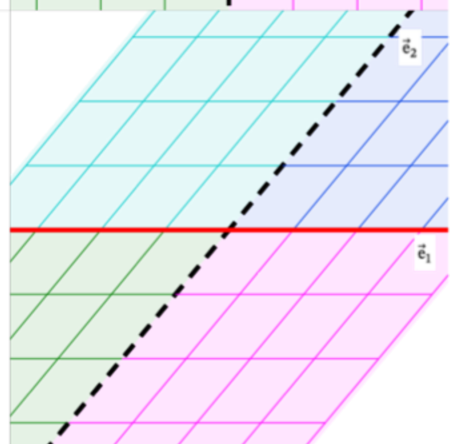
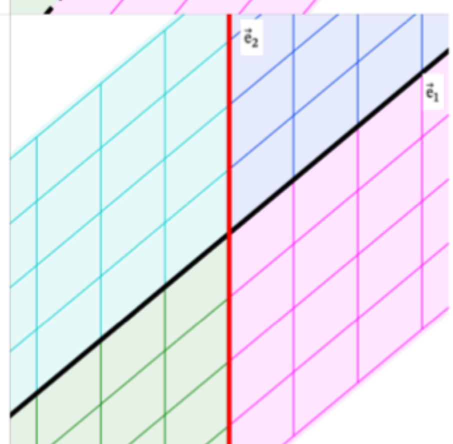
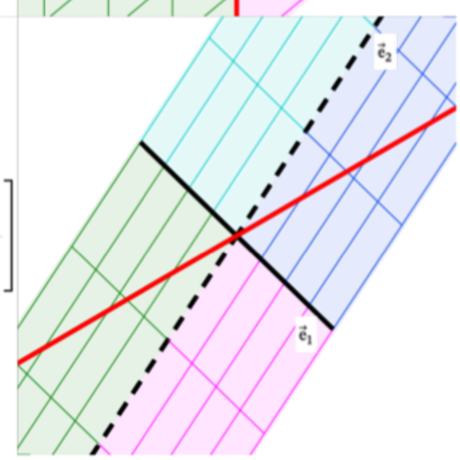
$$A^t = C\, J^t\, C^{-1} \text{ and } J^t = \begin{bmatrix} \lambda^t & t\,\lambda^{t-1} \\ 0 & \lambda^t \end{bmatrix}$$

In contrast, uniform scaling does not require a special basis: $A = S = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

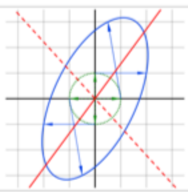
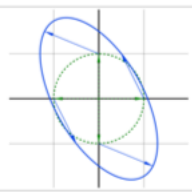
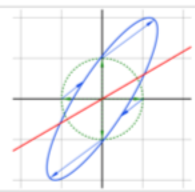
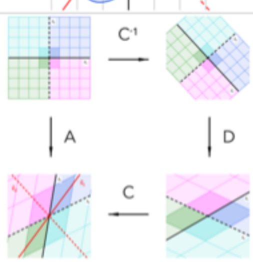
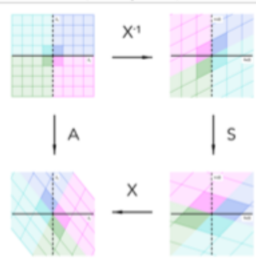
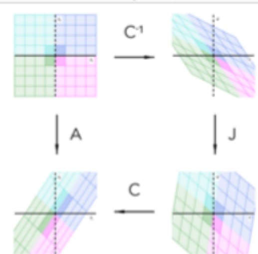
Choosing a basis by matrix type

	Uniform scaling	Scaling–shear	Two distinct eigenvalues
Eigenvalues	One repeated λ	One repeated λ	$\lambda_1 \neq \lambda_2$
Middle matrix degrees of freedom	$S = \lambda I$ fixed	J or J^T depending on basis vector order	$D = \mathrm{diag}(\lambda_1, \lambda_2)$ fixed up to eigenvalue order
Basis-vector directions	Any two independent directions or the standard basis	• One eigendirection is fixed • $\vec{w}$ may be any direction outside it (orthogonal if desired)	Both eigendirections are fixed
Basis-vector scaling	Scale both vectors independently	Choose the scale of $\vec{w}$ freely then set $\vec{v} = (A - \lambda I)\,\vec{w}$	Scale both eigenvectors independently
Degrees of freedom in C	Any invertible C	• One arbitrary direction for $\vec{w}$ outside the eigendirection • One arbitrary scale for $\vec{w}$ • $\vec{v} = (A - \lambda I)\,\vec{w}$ is then fixed	• Two eigendirections are fixed • Scale each eigenvector independently

Matrices with repeated λ: table

	J	C	$A = C\,J\,C^{-1}$	Transformation by A
Uniform scaling • single repeated $\lambda = \vec{a}_{11} = \vec{a}_{22}$ • every nonzero vector is an eigenvector • unchanged with $C\,A\,C^{-1}$	$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1.2 & 0 \\ 0 & 1.2 \end{bmatrix}$	
Jordan (shear-scaling) block • single repeated $\lambda = \vec{a}_{11} = \vec{a}_{22}$ • one eigenvector direction: $\vec{e}_1$	$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1.2 & 1 \\ 0 & 1.2 \end{bmatrix}$	
Transposed Jordan (shear-scaling) block • single repeated $\lambda = \vec{a}_{11} = \vec{a}_{22}$ • one eigenvector direction: $\vec{e}_2$	$\begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.2 & 0 \\ 1 & 1.2 \end{bmatrix}$	
Similarity transformation $M = C\,J\,C^{-1}$ • single repeated λ • one eigenvector direction: $C\,\vec{e}_1$ • generalized eigenvector: $C\,\vec{e}_2$ • $(M - \lambda\,I)\,C\,\vec{e}_2 = C\,\vec{e}_1$	$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} 1.2 & 0.8 \\ 0.7 & 1.4 \end{bmatrix}$	$\begin{bmatrix} 0.45 & \approx 1.286 \\ \approx -0.438 & 1.95 \end{bmatrix}$	

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$	Diagonalization $A = C D C^{-1}$	Rotation-scaling factorization $A = X S X^{-1}$	Shear-scaling factorization $A = C J C^{-1}$
Definition	• $A \vec{v}_1 = \lambda_1 \vec{v}_1$ • $A \vec{v}_2 = \lambda_2 \vec{v}_2$	• $A \vec{x}_1 = \lambda_1 \vec{x}_1$ • $A \vec{x}_2 = \lambda_2 \vec{x}_2$	• $A \vec{v} = \lambda \vec{v}$ • generalized eigenvector $\vec{w}$ (not an eigenvector): $(A - \lambda I) \vec{w} = \vec{v}$
Condition	$(a + d)^2 - 4 (ad - bc) > 0$	$(a + d)^2 - 4 (ad - bc) < 0$	• $(a + d)^2 - 4 (ad - bc) = 0$ • $b \neq 0$ or $c \neq 0$ (non-uniform scaling) • $B = A - \lambda I$ • $B \neq 0$, $B^2 = 0$
λ	$\lambda_1, \lambda_2 \in \mathbb{R}$ $\lambda_1 \neq \lambda_2$	$\lambda_1, \lambda_2 = \alpha \pm \beta i$	$\lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$
Basis vectors	$\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$	$\vec{x}_1, \vec{x}_2 \in \mathbb{C}^2$ $\vec{x}_1 = \mathrm{Re} + i\,\mathrm{Im}$ $\vec{x}_2 = \mathrm{Re} - i\,\mathrm{Im}$ $\mathrm{Re}, \mathrm{Im} \in \mathbb{R}^2$	$\vec{v}, \vec{w} \in \mathbb{R}^2$ $\vec{v} = (A - \lambda I) \vec{w}$ $\vec{v}$ and $\vec{w}$ are linearly independent
Expanding the definitions	$A \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	$A (\mathrm{Re} + i\,\mathrm{Im}) = (\alpha + i\,\beta) (\mathrm{Re} + i\,\mathrm{Im})$ ↓ $A\,\mathrm{Re} + i\,A\,\mathrm{Im} = \alpha\,\mathrm{Re} + i^2\,\beta\,\mathrm{Im} + i\,\beta\,\mathrm{Re} + \alpha\,i\,\mathrm{Im}$ ↓ $A\,\mathrm{Re} + i\,A\,\mathrm{Im} = \alpha\,\mathrm{Re} - \beta\,\mathrm{Im} + i\,(\beta\,\mathrm{Re} + \alpha\,\mathrm{Im})$ ↓ • $A\,\mathrm{Re} = \alpha\,\mathrm{Re} - \beta\,\mathrm{Im}$ • $A\,\mathrm{Im} = \beta\,\mathrm{Re} + \alpha\,\mathrm{Im}$ ↓ $A \begin{bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{bmatrix} = \begin{bmatrix} A\,\mathrm{Re} \\ A\,\mathrm{Im} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{bmatrix}$ Transpose the block equation: $\begin{bmatrix} A\,\mathrm{Re} & A\,\mathrm{Im} \end{bmatrix} = \begin{bmatrix} \mathrm{Re} & \mathrm{Im} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ $A \begin{bmatrix} \mathrm{Re} & \mathrm{Im} \end{bmatrix} = \begin{bmatrix} \mathrm{Re} & \mathrm{Im} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$	$A (\mathrm{Re} + i\,\mathrm{Im}) = (\alpha + i\,\beta) (\mathrm{Re} + i\,\mathrm{Im})$ ↓ $A\,\mathrm{Re} + i\,A\,\mathrm{Im} = \alpha\,\mathrm{Re} + i^2\,\beta\,\mathrm{Im} + i\,\beta\,\mathrm{Re} + \alpha\,i\,\mathrm{Im}$ ↓ $A\,\mathrm{Re} + i\,A\,\mathrm{Im} = \alpha\,\mathrm{Re} - \beta\,\mathrm{Im} + i\,(\beta\,\mathrm{Re} + \alpha\,\mathrm{Im})$ ↓ • $A\,\mathrm{Re} = \alpha\,\mathrm{Re} - \beta\,\mathrm{Im}$ • $A\,\mathrm{Im} = \beta\,\mathrm{Re} + \alpha\,\mathrm{Im}$ ↓ $A \begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix} = \begin{bmatrix} A\vec{v} \\ A\vec{w} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix} \begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix}$ Transpose the block equation: $\begin{bmatrix} A\vec{v} & A\vec{w} \end{bmatrix} = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $A \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$
Middle matrix	$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	$S = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$	Canonical $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$
Middle matrix powers	$D^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$	$r = \mathrm{sqrt}(\alpha^2 + \beta^2)$ $\cos\theta = \alpha/r$, $\sin\theta = \beta/r$ ↓ $S^n = r^n \begin{bmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{bmatrix}$	$J^n = \begin{bmatrix} \lambda^n & n\,\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$
Middle matrix exponentials	$\exp(tD) = \begin{bmatrix} e^{(t\lambda_1)} & 0 \\ 0 & e^{(t\lambda_2)} \end{bmatrix}$	$\exp(tS) = e^{(\alpha t)} \begin{bmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{bmatrix}$	$\exp(tJ) = e^{(\lambda t)} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$
Middle matrix: • geometric action • determinant	• scales along $\vec{v}_1$ and $\vec{v}_2$ • $\det(D) = \lambda_1 \lambda_2$	• rotates and scales in the [Re | Im] basis • $\det(S) = \alpha^2 + \beta^2 = \lvert \lambda \rvert^2$	• maps $\vec{v} \to \lambda \vec{v}$ and $\vec{w} \to \vec{v} + \lambda \vec{w}$ • $\det(J) = \lambda^2$
Basis matrix	$C = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$	$X = \begin{bmatrix} \mathrm{Re} & \mathrm{Im} \end{bmatrix}$	$C = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$
	If A can be diagonalized using an orthogonal basis, we may scale the basis vectors	If Re and Im are orthogonal and have equal length,	Let $B = A - \lambda I$ For a defective repeated-λ matrix, $B^2 = 0$ and $\mathrm{rank}(B) = 1$ Choose a unit eigenvector $\vec{v}$ and choose a unit vector $\vec{w}$ orthogonal to $\vec{v}$ ↓ $C = [\, \vec{v} \mid \vec{w} \,]$ is orthogonal Since $B \vec{v} = \vec{0}$ and $\mathrm{col}(B) = \mathrm{null}(B) = \mathrm{span}(\vec{v})$,

Case where the basis vectors are orthogonal	so that C is orthogonal ↓ $C^{-1} = C^T$ ↓ $A = C D C^T$ ↓ $A^T = (C D C^T)^T = C D C^T = A$ ↓ orthogonal diagonalization requires $A = A^T$	we may write $X = r\,Q$ with Q orthogonal ↓ $X^{-1} = (1/r)\,Q^T$ ↓ $A = X S X^{-1} = (r\,Q)\,S\,((1/r)\,Q^T)$ $= Q S Q^T$ ↓ A is rotation-scaling of the standard basis	the vector $B\vec{w}$ must be a scalar multiple of $\vec{v}$ ↓ $B\vec{w} = s\,\vec{v}$ $B \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} = \begin{bmatrix} B\vec{v} & B\vec{w} \end{bmatrix} = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix}$ Therefore $C^T B C$ = [[0, s], [0, 0]] and $C^T A C$ = [[λ, s], [0, λ]] ↓ canonical J with $J_{12} = 1$ is obtained iff $\lvert s \rvert = 1$ Equivalently, for the original matrix A, canonical J with orthogonal C occurs iff $\lVert A - \lambda I \rVert_F = 1$ (for the 2×2 defective case, equivalently $\lvert b - c \rvert = 1$)
A^n	$A^n = C D^n C^{-1}$	$A^n = X S^n X^{-1}$	$A^n = C J^n C^{-1}$
exp(tA)	$\exp(tA) = C \exp(tD) C^{-1}$	$\exp(tA) = X \exp(tS) X^{-1}$	$\exp(tA) = C \exp(tJ) C^{-1}$
Determinant consequence	$A = C D C^{-1}$ and $\det(C)\det(C^{-1}) = 1$ ↓ $\det(A) = \det(D)$ ↓ $\det(A) = \lambda_1 \lambda_2$	$A = X S X^{-1}$ and $\det(X)\det(X^{-1}) = 1$ ↓ $\det(A) = \det(S)$ ↓ $\det(A) = \alpha^2 + \beta^2$ ↓ $\det(A) = \lvert \lambda \rvert^2$	$A = C J C^{-1}$ and $\det(C)\det(C^{-1}) = 1$ ↓ $\det(A) = \det(J)$ ↓ $\det(A) = \lambda^2$
Sequence of transformations	• Move to [$\vec{v}_1 \mid \vec{v}_2$] basis • Scale • Move back	• Move to [Re | Im] basis • Rotate and scale • Move back	• Move to [$\vec{v} \mid \vec{w}$] basis • Shear and scale • Move back
Numerical example	$A = \begin{bmatrix} 0.5 & 2 \\ 3 & 1 \end{bmatrix} =$ $\begin{bmatrix} \approx 0.59 & \approx -0.67 \\ \approx 0.8 & \approx 0.74 \end{bmatrix} \begin{bmatrix} \approx 3.21 & 0 \\ 0 & \approx -1.71 \end{bmatrix} \begin{bmatrix} \approx 0.76 & \approx 0.68 \\ \approx -0.82 & \approx 0.61 \end{bmatrix}$	$A = \begin{bmatrix} 0.5 & -1.2 \\ 1 & 1.5 \end{bmatrix} =$ $\begin{bmatrix} \approx -0.34 & \approx 0.66 \\ \approx 0.67 & 0 \end{bmatrix} \begin{bmatrix} 1 & \approx 0.97 \\ \approx -0.97 & 1 \end{bmatrix} \begin{bmatrix} 0 & \approx 1.48 \\ \approx 1.52 & \approx 0.76 \end{bmatrix}$	$A = \begin{bmatrix} 0.45 & \approx 1.29 \\ \approx -0.44 & 1.95 \end{bmatrix} =$ $\begin{bmatrix} 1.2 & 0.8 \\ 0.7 & 1.4 \end{bmatrix} \begin{bmatrix} 1.2 & 1 \\ 0 & 1.2 \end{bmatrix} \begin{bmatrix} 1.25 & \approx -0.71 \\ \approx -0.63 & \approx 1.07 \end{bmatrix}$
selected $\hat{v} \rightarrow A\hat{v}$ dominant $\vec{v}$ other $\vec{v}$ all $\hat{v}$ all $A\hat{v}$			
Sequence of transformations			
Factorize A A is given: compute the factors	• Compute λ_1, λ_2 from $\det(A - \lambda I) = 0$ • Compute $\vec{v}_1, \vec{v}_2$ from $(A - \lambda I)\,\vec{v} = \vec{0}$ • Choose the scaling of $\vec{v}_1$ and $\vec{v}_2$ independently	• Compute $\lambda_{1,2} = \alpha \pm \beta\, i$ from $\det(A - \lambda I) = 0$ • Using either λ_1 or λ_2, compute one $\vec{x}$ from $(A - \lambda I)\,\vec{x} = \vec{0}$ • Choose the scaling of $\vec{x}$ • Split $\vec{x}$ into Re and Im	• Compute the repeated λ from $\det(A - \lambda I) = 0$ • Set $B = A - \lambda I$ • Find the eigendirection from $B\vec{v} = \vec{0}$ • Choose $\vec{w}$ outside it (may choose $\vec{w}$ orthogonal to it) • Scale $\vec{w}$ arbitrarily • Set $\vec{v} = B\vec{w}$ • Build C = [$\vec{v} \mid \vec{w}$] and use the canonical J
Construct A A is not given: choose the factors, then build A	• Choose λ_1, λ_2 ($\lambda_1 \neq \lambda_2$) • Choose independent $\vec{v}_1, \vec{v}_2$ • Build C and D • Set $A = C D C^{-1}$	• Choose α, β with β ≠ 0 • Choose independent Re and Im • Build X and S • Set $A = X S X^{-1}$	• Choose λ • Choose the desired eigendirection • Choose $\vec{w}$ outside it (orthogonal if desired) • Choose a nonzero $\vec{v}$ on the eigendirection • Build C = [$\vec{v} \mid \vec{w}$] • Use the canonical Jordan block J • Set $A = C J C^{-1}$