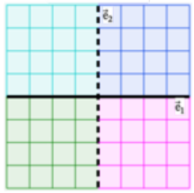
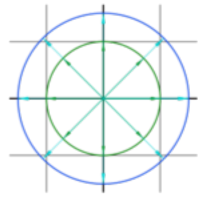
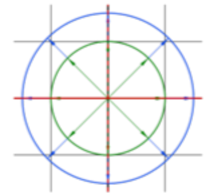
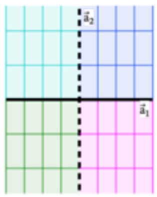
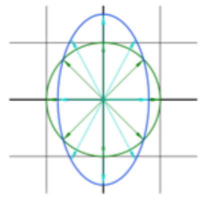
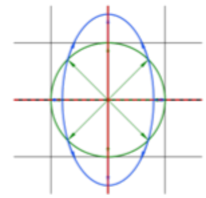
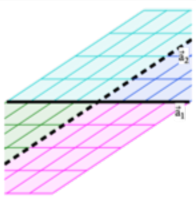
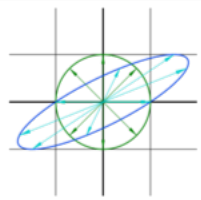
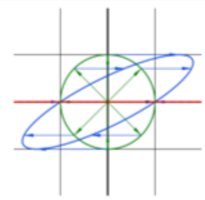
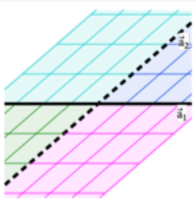
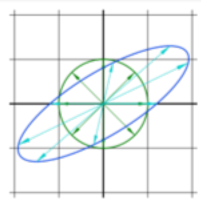
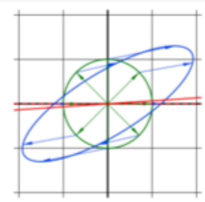
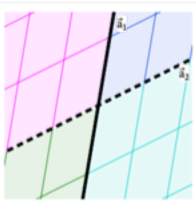
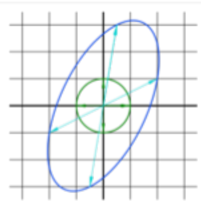
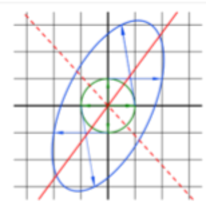
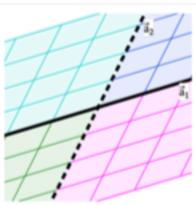
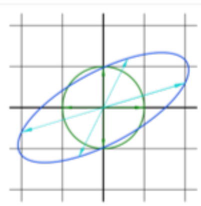
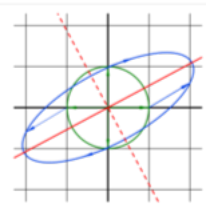
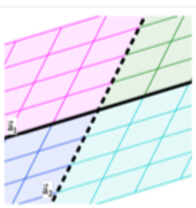
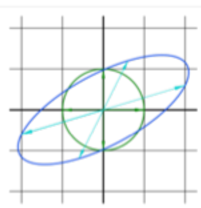
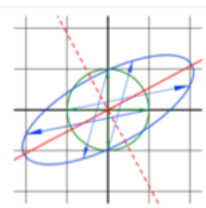
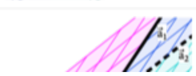
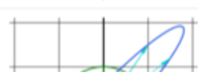
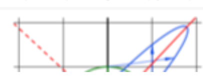
Definition

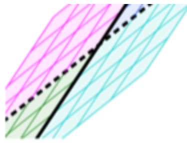
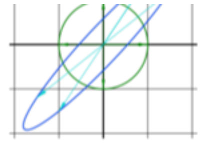
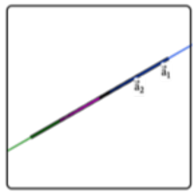
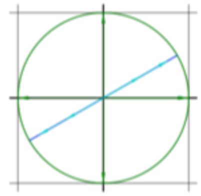
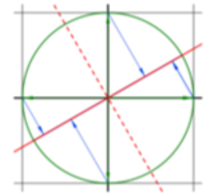
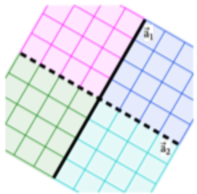
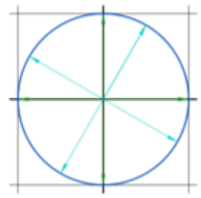
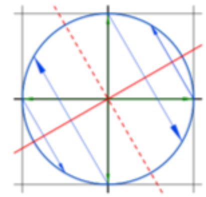
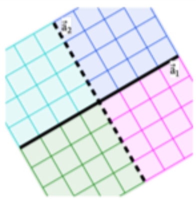
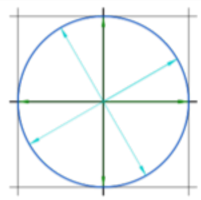
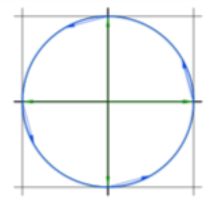
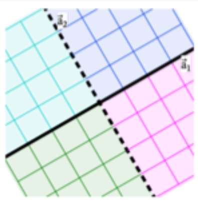
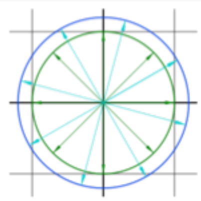
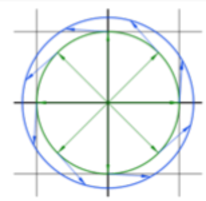
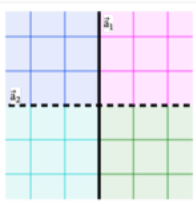
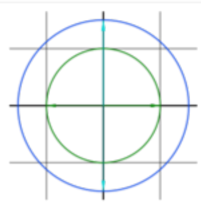
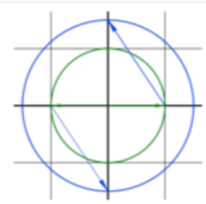
An eigenvector of a matrix A is a non-0 vector that
stays on the same line after multiplication by A:

$$A \vec{x} = \lambda \vec{x}$$

λ is a scalar, real (positive, negative, or 0) or complex valued

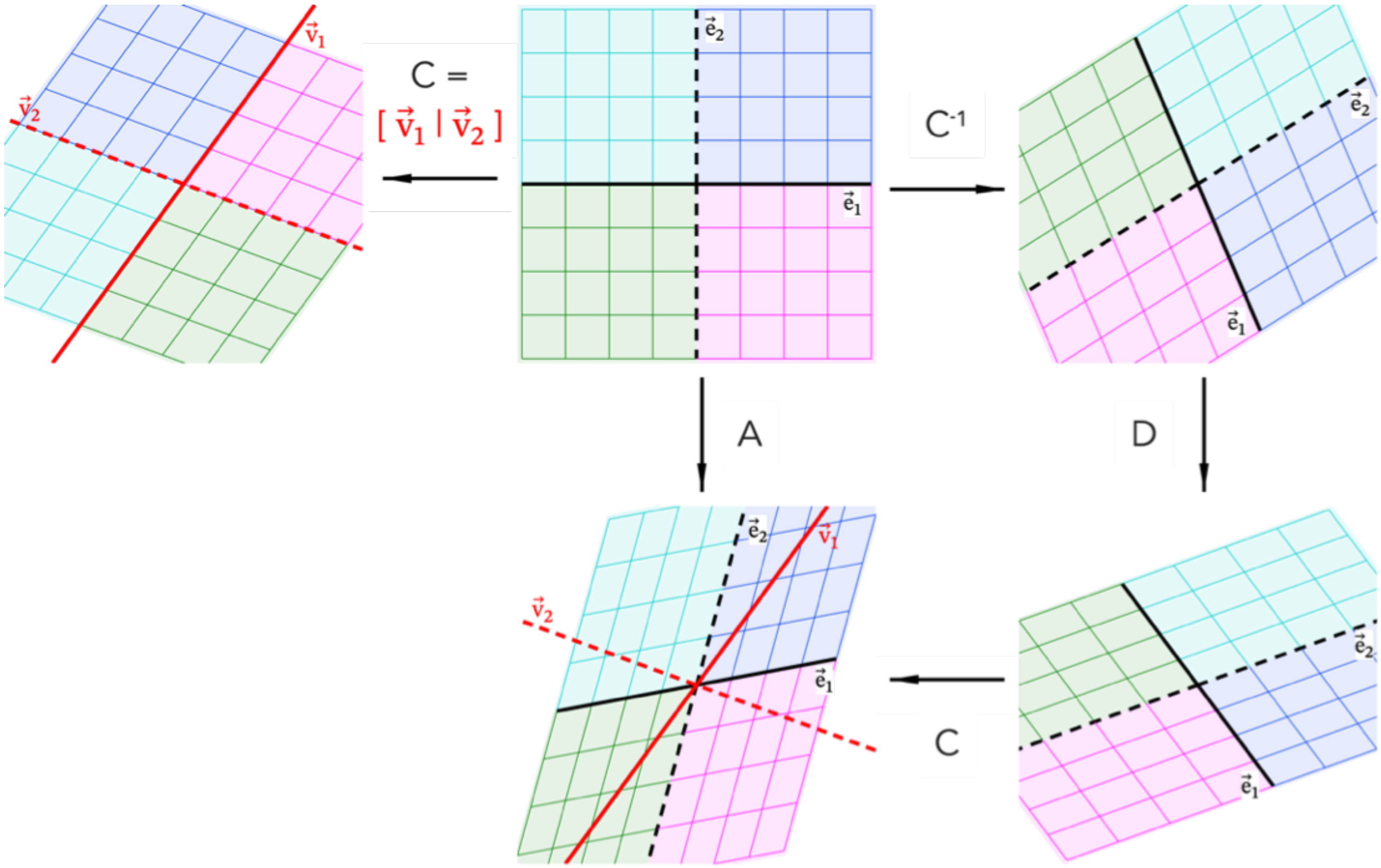
Examples:

Matrix	Transformation description	Starting with identity grid 	λ & $\vec{x}$ satisfying $A\vec{x} = \lambda\vec{x}$	○ all $\hat{v}$ ↗ selected $\hat{v}$ ⬭ all $A\hat{v}$ ↗ selected $A\hat{v}$	↗ selected $\hat{v} \rightarrow A\hat{v}$ — dominant $\vec{x}$ - - - other $\vec{x}$
Uniform scaling $\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$	unit circle ↓ circle		$A\vec{x} = \lambda\vec{x}$ for every nonzero $\vec{x}$ $\lambda = A_{11} = A_{22}$		
Diagonal, non-uniform scaling $\begin{bmatrix} 0.8 & 0 \\ 0 & 1.5 \end{bmatrix}$	unit circle ↓ axis-aligned ellipse		$A\vec{e}_1 = \lambda_1\vec{e}_1$ $A\vec{e}_2 = \lambda_2\vec{e}_2$ ↓ $\vec{x}_1 = \vec{e}_1$ $\vec{x}_2 = \vec{e}_2$ $\lambda_1 = A_{11}$ $\lambda_2 = A_{22}$		
Shear $\begin{bmatrix} 1 & 1.5 \\ 0 & 1 \end{bmatrix}$	unit circle ↓ tilted ellipse $\vec{e}_2$ slides along $\vec{e}_1$		only one $\vec{x}$ and one λ: $A\vec{e}_1 = \lambda\vec{e}_1$ ↓ $\vec{x} = \vec{e}_1$ $\lambda = A_{11} = A_{22}$		
Upper-triangular $\begin{bmatrix} 1.2 & 1.5 \\ 0 & 1.3 \end{bmatrix}$	unit circle ↓ tilted ellipse		$\lambda_1 = A_{11}$ $\lambda_2 = A_{22}$ $\vec{x}_1$ is multiple of $\vec{e}_1$ $\vec{x}_2$ is usually not $\vec{e}_2$		
General non-symmetric $\begin{bmatrix} 0.5 & 2 \\ 3 & 1 \end{bmatrix}$	unit circle ↓ tilted ellipse ellipse axes not aligned with $\vec{x}_1$ and $\vec{x}_2$		for this example: $\lambda_1, \lambda_2 \in \mathbb{R}$ two $\vec{x} \in \mathbb{R}^2$ $\vec{x}_1 \cdot \vec{x}_2 \neq 0$ ellipse axes are not $\vec{x}$ directions		
Symmetric $\lambda_1, \lambda_2 > 0$ $\begin{bmatrix} 2 & 0.6 \\ 0.6 & 1.2 \end{bmatrix}$	unit circle ↓ tilted ellipse ellipse axes align with $\vec{x}_1$ and $\vec{x}_2$ both directions stretched		two $\lambda \in \mathbb{R}$ two $\vec{x} \in \mathbb{R}^2$ $\lambda_1 > 0$ $\lambda_2 > 0$ $\vec{x}_1 \cdot \vec{x}_2 = 0$		
Symmetric $\lambda_1, \lambda_2 < 0$ $\begin{bmatrix} -2 & -0.6 \\ -0.6 & -1.2 \end{bmatrix}$	unit circle ↓ tilted ellipse ellipse axes align with $\vec{x}_1$ and $\vec{x}_2$ both directions flipped		two $\lambda \in \mathbb{R}$ two $\vec{x} \in \mathbb{R}^2$ $\lambda_1 < 0$ $\lambda_2 < 0$ $\vec{x}_1 \cdot \vec{x}_2 = 0$		
Symmetric mixed signs	unit circle ↓		two $\lambda \in \mathbb{R}$		

mixed signs $\begin{bmatrix} 1 & 1.5 \\ 1.5 & 1.2 \end{bmatrix}$	↓ tilted ellipse ellipse axes align with $\vec{x}_1$ and $\vec{x}_2$ one direction flips		two $\vec{x} \in \mathbb{R}^2$ $\lambda_1 > 0$ $\lambda_2 < 0$ $\vec{x}_1 \cdot \vec{x}_2 = 0$		
Projection also symmetric $\begin{bmatrix} 0.75 & \approx 0.433 \\ \approx 0.433 & 0.25 \end{bmatrix}$	unit circle ↓ line segment		$\vec{x}_1$ = image direction $\lambda_1 = 1$ $\vec{x}_2$ = collapsed direction $\lambda_2 = 0$ $\vec{x}_1 \cdot \vec{x}_2 = 0$		
Reflection also symmetric $\begin{bmatrix} 0.5 & \approx 0.866 \\ \approx 0.866 & -0.5 \end{bmatrix}$	unit circle ↓ unit circle flips across a line		$\vec{x}_1$ = mirror direction $\lambda_1 = 1$ $\vec{x}_2$ = orthogonal direction $\lambda_2 = -1$ $\vec{x}_1 \cdot \vec{x}_2 = 0$		
Rotation $\begin{bmatrix} \approx 0.866 & -0.5 \\ 0.5 & \approx 0.866 \end{bmatrix}$	unit circle ↓ unit circle		no $\vec{x} \in \mathbb{R}^2$ satisfies $A\vec{x} = \lambda\vec{x}$		
Rotation-scaling $\begin{bmatrix} \approx 1.039 & -0.6 \\ 0.6 & \approx 1.039 \end{bmatrix}$	unit circle ↓ scaled circle		no $\vec{x} \in \mathbb{R}^2$ satisfies $A\vec{x} = \lambda\vec{x}$		
Skew-symmetric $\begin{bmatrix} 0 & -1.5 \\ 1.5 & 0 \end{bmatrix}$	unit circle ↓ scaled circle quarter-turn + scale		no $\vec{x} \in \mathbb{R}^2$ satisfies $A\vec{x} = \lambda\vec{x}$ more: $A\vec{x} \perp \vec{x}$ for every $\vec{x} \in \mathbb{R}^2$		

Diagonalization of A: $A = C D C^{-1}$

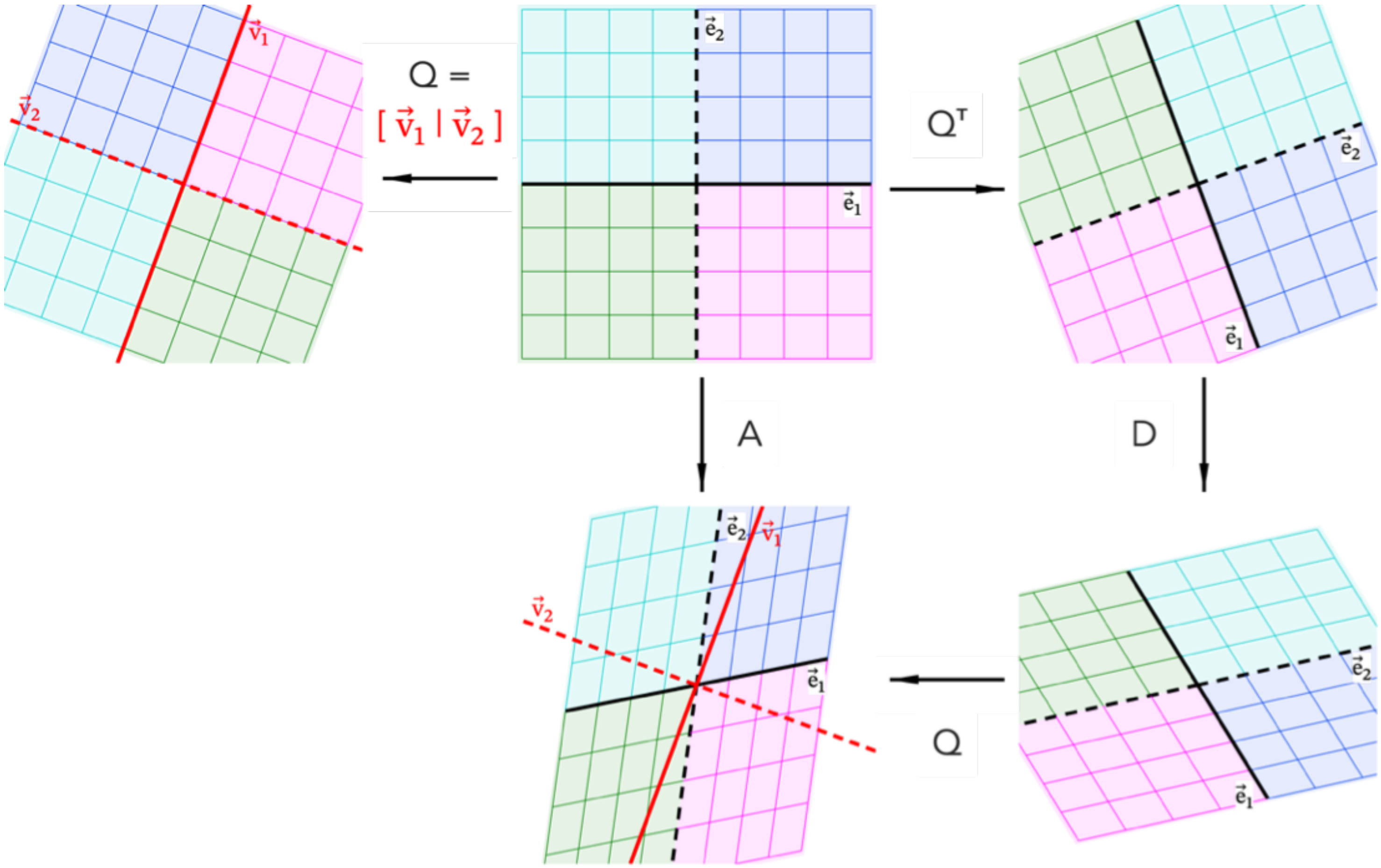
$$\begin{bmatrix} 0.8 & 0.3 \\ 0.15 & 1.1 \end{bmatrix} = \begin{bmatrix} \approx 0.591 & \approx -0.939 \\ \approx 0.807 & \approx 0.344 \end{bmatrix} \begin{bmatrix} \approx 1.21 & 0 \\ 0 & \approx 0.69 \end{bmatrix} \begin{bmatrix} \approx 0.358 & \approx 0.977 \\ \approx -0.84 & \approx 0.615 \end{bmatrix}$$



Orthogonal diagonalization of $A = A^T$

$A = Q\,D\,Q^T,\ Q^{-1} = Q^T$

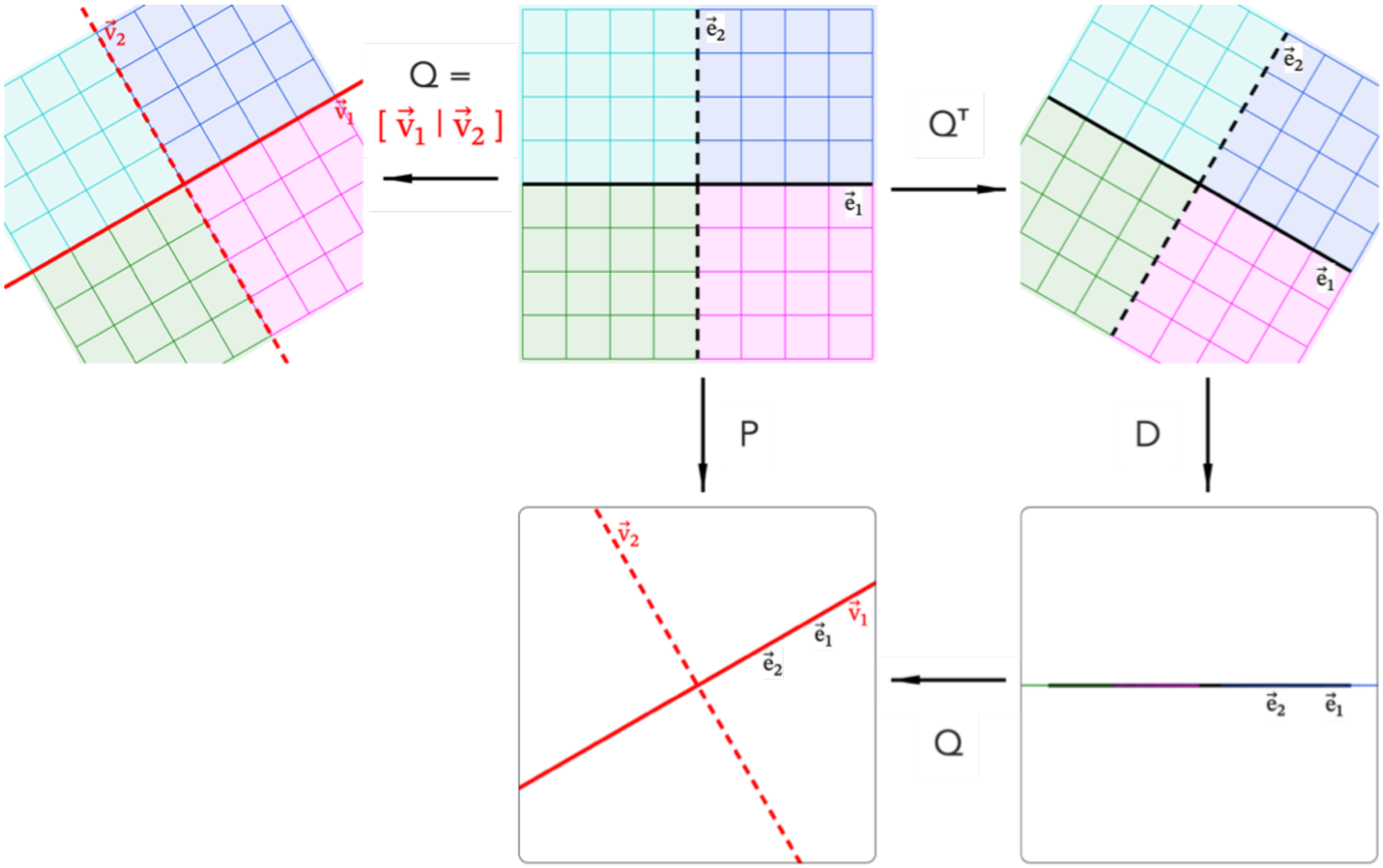
$$\begin{bmatrix} 0.75 & 0.15 \\ 0.15 & 1.1 \end{bmatrix} = \begin{bmatrix} \approx 0.347 & \approx -0.938 \\ \approx 0.938 & \approx 0.347 \end{bmatrix} \begin{bmatrix} \approx 1.155 & 0 \\ 0 & \approx 0.695 \end{bmatrix} \begin{bmatrix} \approx 0.347 & \approx 0.938 \\ \approx -0.938 & \approx 0.347 \end{bmatrix}$$



Diagonalization of orthogonal projection matrix P: $P^2 = P$

$P = Q\,D\,Q^T$, $(P = P^T)$

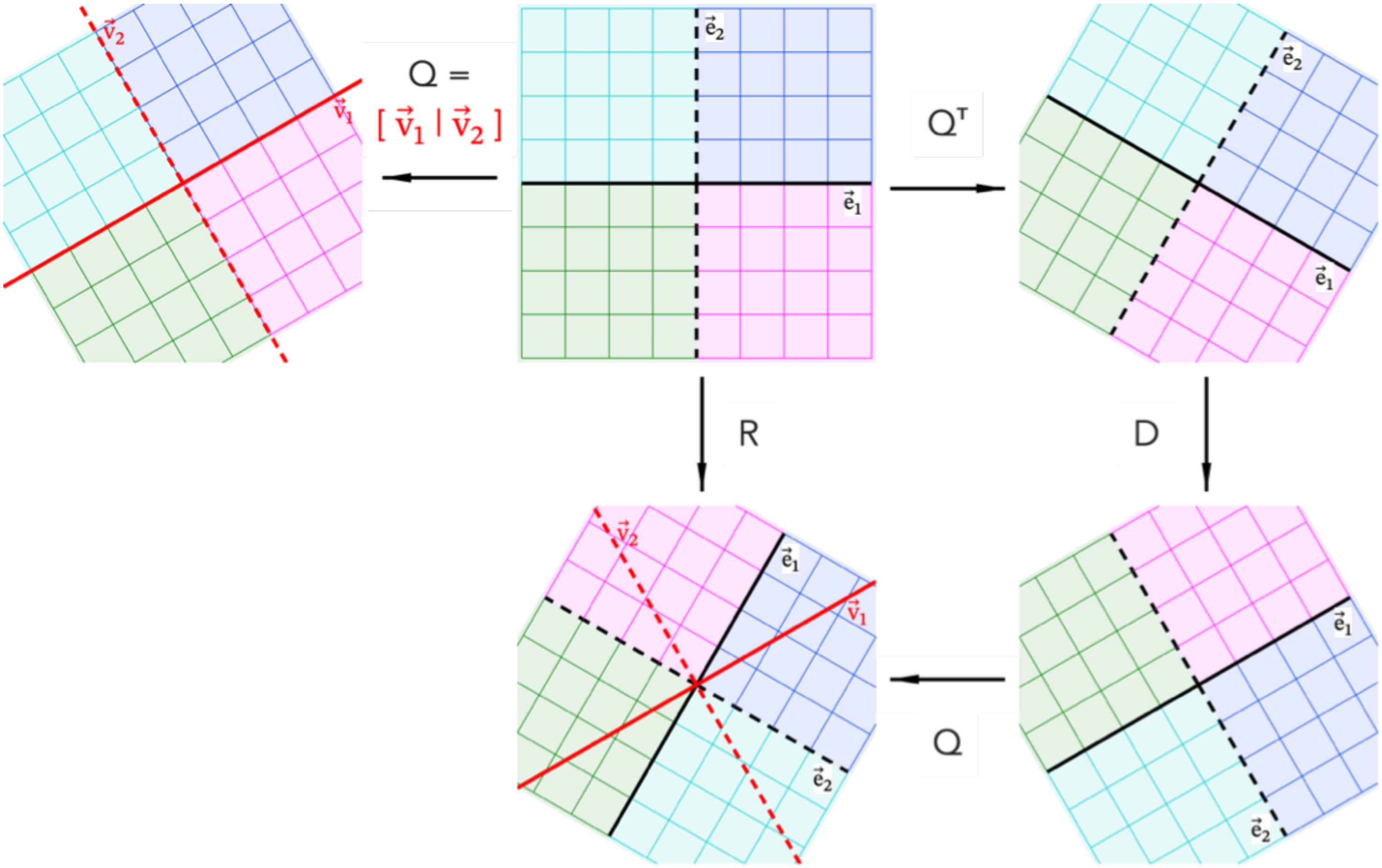
$$\begin{bmatrix} 0.75 & \approx 0.433 \\ \approx 0.433 & 0.25 \end{bmatrix} = \begin{bmatrix} \approx 0.866 & -0.5 \\ 0.5 & \approx 0.866 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \approx 0.866 & 0.5 \\ -0.5 & \approx 0.866 \end{bmatrix}$$



Diagonalization of reflection (involution) matrix R: $R^2 = I$

$R = Q\,D\,Q^T$, $(R = R^T)$

$$\begin{bmatrix} 0.5 & \approx 0.866 \\ \approx 0.866 & -0.5 \end{bmatrix} = \begin{bmatrix} \approx 0.866 & -0.5 \\ 0.5 & \approx 0.866 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \approx 0.866 & 0.5 \\ -0.5 & \approx 0.866 \end{bmatrix}$$



Interpolating the matrices for animations

We move gradually from the identity matrix to A
For $0 \le t \le 1$, define

$$M(t) = (1 - t)\, I + t\, A$$

At $t = 0$, $M(t) = I$
At $t = 1$, $M(t) = A$

If $\vec{x}$ is an eigenvector of A, its direction stays fixed throughout:

$$A \vec{x} = \lambda \vec{x}$$

$$M(t) \vec{x} = \big((1 - t) I + t A\big) \vec{x}$$

$$M(t) \vec{x} = (1 - t) \vec{x} + t \lambda \vec{x}$$

$$M(t) \vec{x} = (1 - t + t \lambda) \vec{x}$$

Thus $\vec{x}$ is also an eigenvector of every intermediate matrix M(t)

In contrast, the eigenvalues are not fixed:

$$\lambda(t) = 1 - t + t \lambda = 1 + t (\lambda - 1)$$

↓

- each λ starts at 1 and moves linearly to its final value λ
- if $\lambda < 0$, the direction shrinks to zero, then reappears reversed

the crossing happens at

$$t = \frac{1}{1 - \lambda}$$
