

Uniform scaling

$$A = C D C^{-1}$$

$$\left[\begin{array}{c|c} 1.5 & 0 \\ \hline 0 & 1.5 \end{array} \right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right] \left[\begin{array}{c|c} 1.5 & 0 \\ \hline 0 & 1.5 \end{array} \right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right]$$

$$A \vec{x} = \lambda \vec{x}$$

for every nonzero $\vec{x}$

$$\lambda = 1.5 > 1$$

$$\begin{aligned} \lambda &= A_{11} = A_{22} \\ &= D_{11} = D_{22} \end{aligned}$$

↓

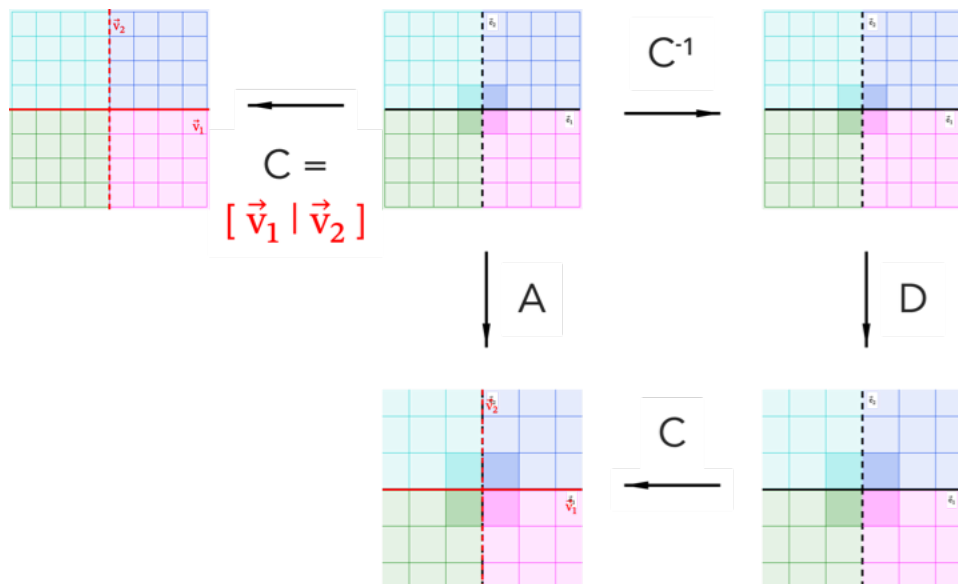
$$C D = D C$$

↓

any invertible C is valid

shown here:

$$C = [\vec{e}_1 | \vec{e}_2] = I$$





Upper-triangular

$$A = C D C^{-1}$$

$$\left[\begin{array}{c|c} 1.1 & 0.9 \\ \hline 0 & 2 \end{array} \right] = \left[\begin{array}{c|c} 1 & \approx 0.707 \\ \hline 0 & \approx 0.707 \end{array} \right] \left[\begin{array}{c|c} 1.1 & 0 \\ \hline 0 & 2 \end{array} \right] \left[\begin{array}{c|c} 1 & -1 \\ \hline 0 & \approx 1.414 \end{array} \right]$$

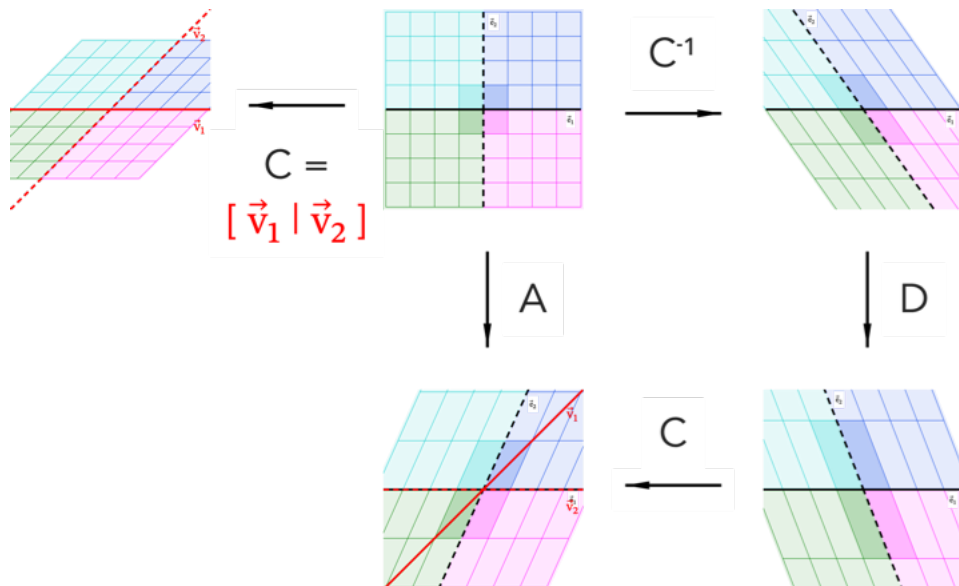
$$\lambda_1 = A_{11} = 1.1 > 1$$

$$\lambda_2 = A_{22} = 2 > 1$$

$\vec{x}_1$ is a multiple of $\vec{e}_1$

$\vec{x}_2$ is usually not $\vec{e}_2$

$$C = [\vec{x}_1 \mid \vec{x}_2]$$



General
non-symmetric
 $A = C D C^{-1}$

$$\left[\begin{array}{c|c} 0.5 & 2 \\ \hline 3 & 1 \end{array} \right] = \left[\begin{array}{c|c} \approx 0.593 & \approx -0.671 \\ \hline \approx 0.805 & \approx 0.742 \end{array} \right] \left[\begin{array}{c|c} \approx 3.212 & 0 \\ \hline 0 & \approx -1.712 \end{array} \right] \left[\begin{array}{c|c} \approx 0.757 & \approx 0.684 \\ \hline \approx -0.821 & \approx 0.606 \end{array} \right]$$

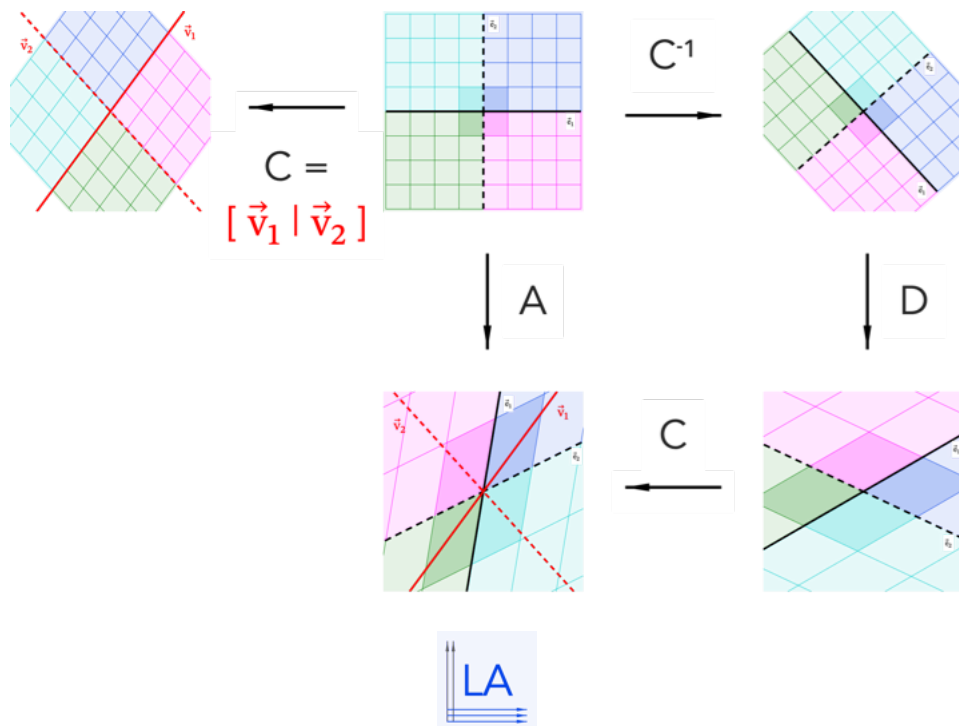
$$\lambda_1 \approx 3.212 > 1$$

$$\lambda_2 \approx -1.712 < -1$$

two $\vec{x} \in \mathbb{R}^2$

$$\vec{x}_1 \cdot \vec{x}_2 \neq 0$$

$$C = [\vec{x}_1 | \vec{x}_2]$$



Symmetric
positive λ
 $A = Q D Q^T$

$$\begin{bmatrix} 2 & 0.6 \\ 0.6 & 1.2 \end{bmatrix} = \begin{bmatrix} \approx 0.882 & \approx -0.472 \\ \approx 0.472 & \approx 0.882 \end{bmatrix} \begin{bmatrix} \approx 2.321 & 0 \\ 0 & \approx 0.879 \end{bmatrix} \begin{bmatrix} \approx 0.882 & \approx 0.472 \\ \approx -0.472 & \approx 0.882 \end{bmatrix}$$

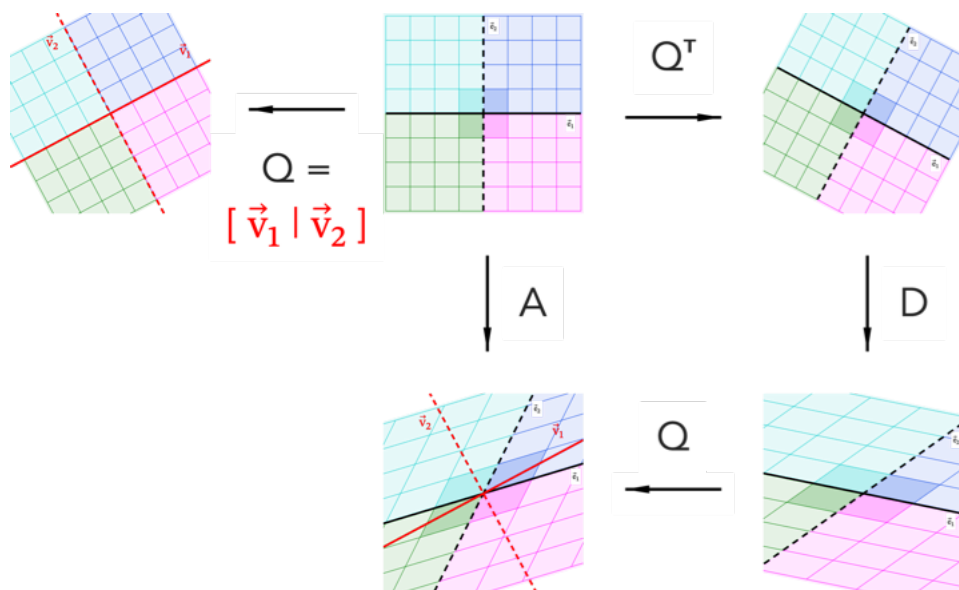
$$\lambda_1 \approx 2.321 > 1$$

$$0 < \lambda_2 \approx 0.879 < 1$$

$$\vec{x}_1 \cdot \vec{x}_2 = 0$$

$\vec{v} \rightarrow A \vec{v}$ arrows cross
neither eigenline

$$Q = [\vec{x}_1 \mid \vec{x}_2]$$





Symmetric
negative λ
 $A = Q D Q^T$

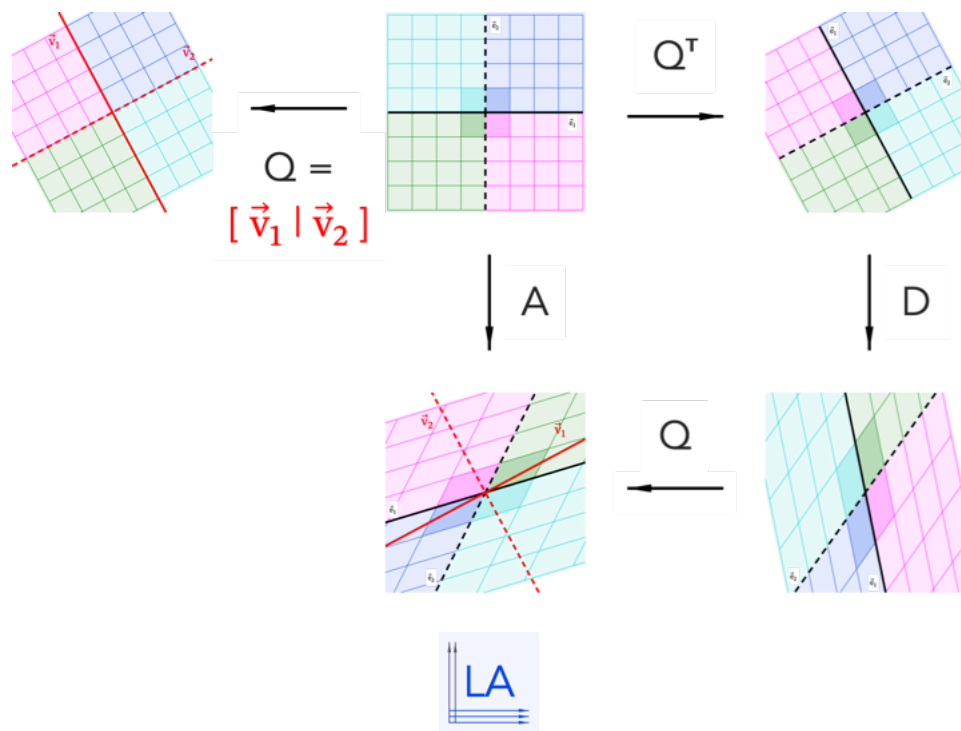
$$\begin{bmatrix} -2 & -0.6 \\ -0.6 & -1.2 \end{bmatrix} = \begin{bmatrix} \approx -0.472 & \approx 0.882 \\ \approx 0.882 & \approx 0.472 \end{bmatrix} \begin{bmatrix} \approx -0.879 & 0 \\ 0 & \approx -2.321 \end{bmatrix} \begin{bmatrix} \approx -0.472 & \approx 0.882 \\ \approx 0.882 & \approx 0.472 \end{bmatrix}$$

$$\begin{aligned} -1 < \lambda_1 &\approx -0.879 < 0 \\ \lambda_2 &\approx -2.321 < -1 \end{aligned}$$

$$\vec{x}_1 \cdot \vec{x}_2 = 0$$

$\vec{v} \rightarrow A \vec{v}$ arrows cross
both eigenlines

$$Q = [\vec{x}_1 \mid \vec{x}_2]$$



Symmetric
mixed-sign λ
 $A = Q D Q^T$

$$\begin{bmatrix} 1 & 1.5 \\ 1.5 & 1.2 \end{bmatrix} = \begin{bmatrix} \approx 0.683 & \approx -0.73 \\ \approx 0.73 & \approx 0.683 \end{bmatrix} \begin{bmatrix} \approx 2.603 & 0 \\ 0 & \approx -0.403 \end{bmatrix} \begin{bmatrix} \approx 0.683 & \approx 0.73 \\ \approx -0.73 & \approx 0.683 \end{bmatrix}$$

$$\lambda_1 \approx 2.603 > 1$$

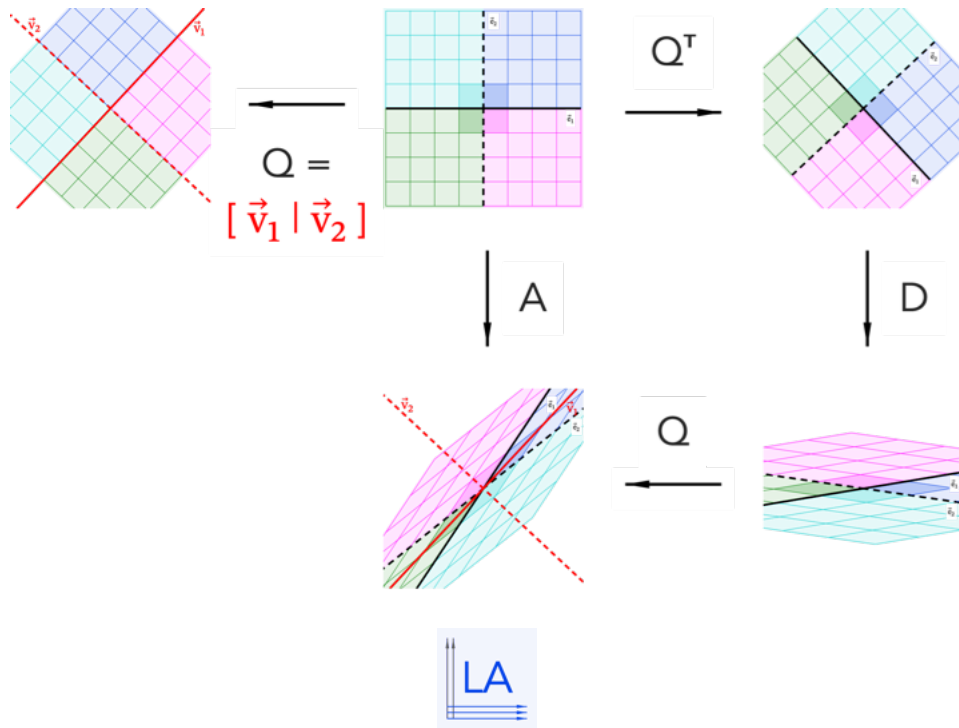
$$-1 < \lambda_2 \approx -0.403 < 0$$

$$\vec{x}_1 \cdot \vec{x}_2 = 0$$

$\vec{v} \rightarrow A \vec{v}$ arrows cross

the positive- λ eigenline

$$Q = [\vec{x}_1 \mid \vec{x}_2]$$



Projection
also symmetric
 $P = Q D Q^T$

$$\begin{bmatrix} 0.75 & \approx 0.433 \\ \approx 0.433 & 0.25 \end{bmatrix} = \begin{bmatrix} \approx 0.866 & -0.5 \\ 0.5 & \approx 0.866 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \approx 0.866 & 0.5 \\ -0.5 & \approx 0.866 \end{bmatrix}$$

$\vec{x}_1 = \text{col}(A)$ direction
 $\lambda_1 = 1$

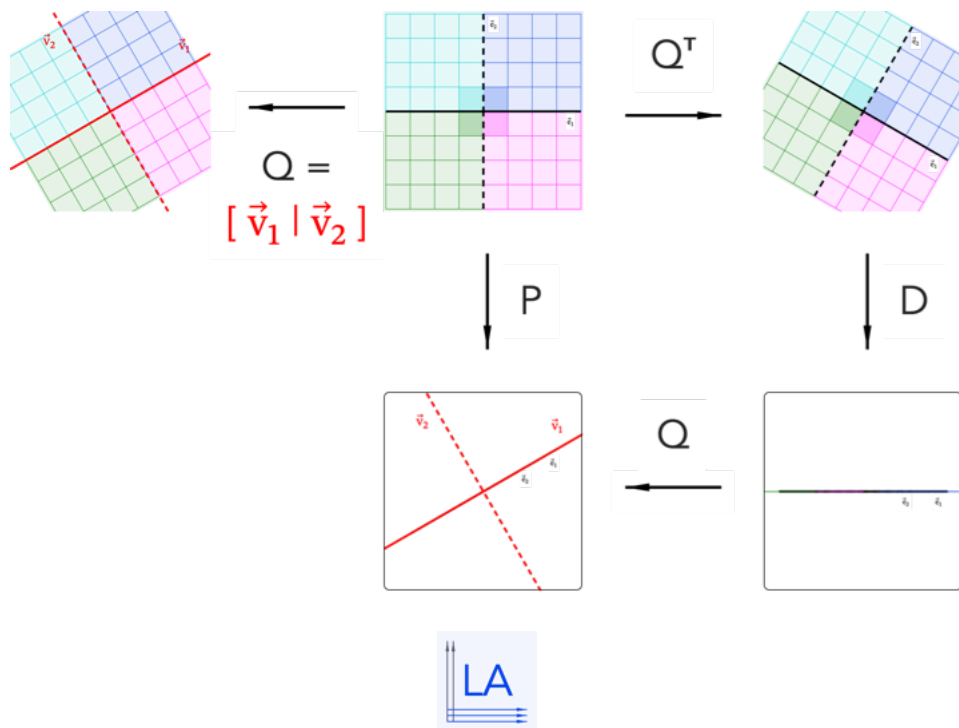
$\vec{x}_2 = \ell\text{-null}(A)$ direction

$$\lambda_2 = 0$$

$$\vec{x}_1 \cdot \vec{x}_2 = 0$$

$\vec{v} \rightarrow A \vec{v}$ arrows end on
the $\lambda = 1$ eigenline

$$Q = [\vec{x}_1 \mid \vec{x}_2]$$



Reflection
also symmetric
 $R = Q D Q^T$

$$\begin{bmatrix} 0.5 & \approx 0.866 \\ \approx 0.866 & -0.5 \end{bmatrix} = \begin{bmatrix} \approx 0.866 & -0.5 \\ 0.5 & \approx 0.866 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \approx 0.866 & 0.5 \\ -0.5 & \approx 0.866 \end{bmatrix}$$

 $\vec{x}_1 = \text{mirror direction}$

$$\lambda_1 = 1$$

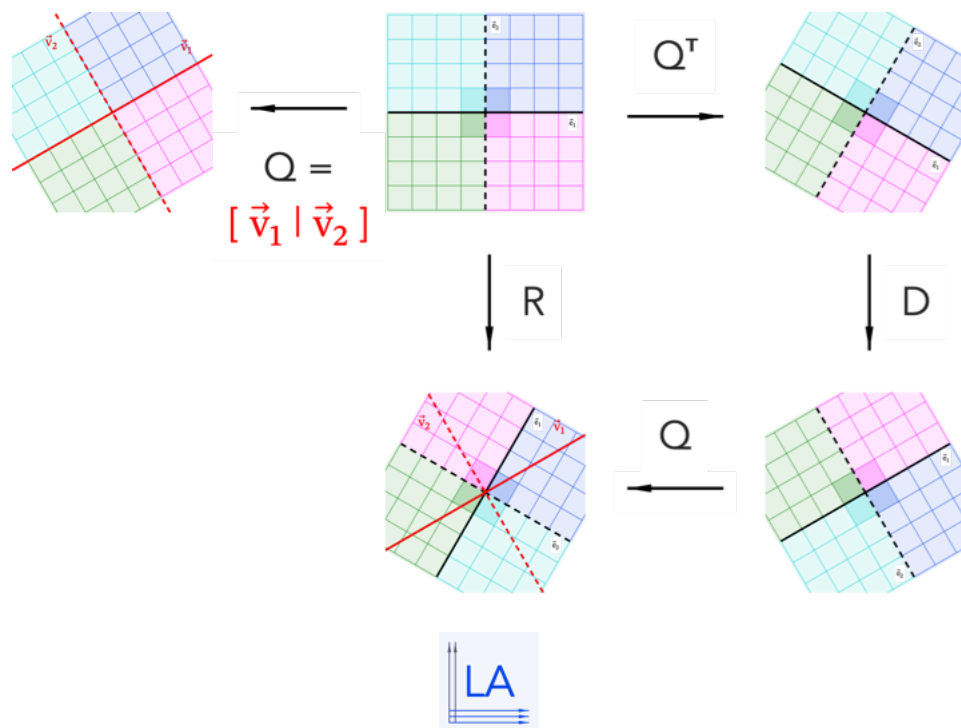
$\vec{x}_2$ = orthogonal direction

$$\lambda_2 = -1$$

$$\vec{x}_1 \cdot \vec{x}_2 = 0$$

$\vec{v} \rightarrow A \vec{v}$ arrows cross
the mirror line

$$Q = [\vec{x}_1 \mid \vec{x}_2]$$



Diagonalizable matrices with $\lambda \in \mathbb{R}$

Uniform scaling

$$A = C D C^{-1}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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for every nonzero $\vec{x}$

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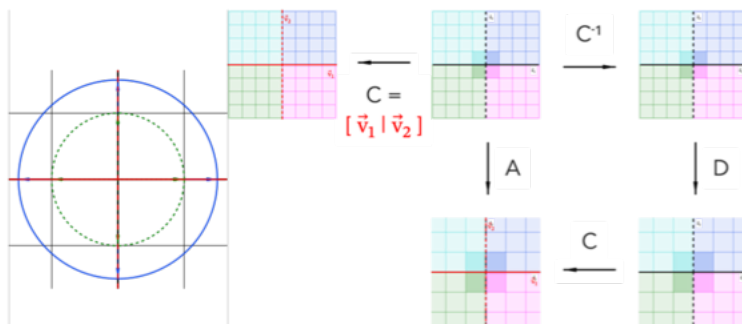
$$= D_{11} = D_{22}$$

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any invertible C is valid

$$\text{shown here:}$$

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Upper-triangular

$$A = C D C^{-1}$$

$$\begin{bmatrix} 1.1 & 0.9 \\ 0 & 2 \end{bmatrix} =$$

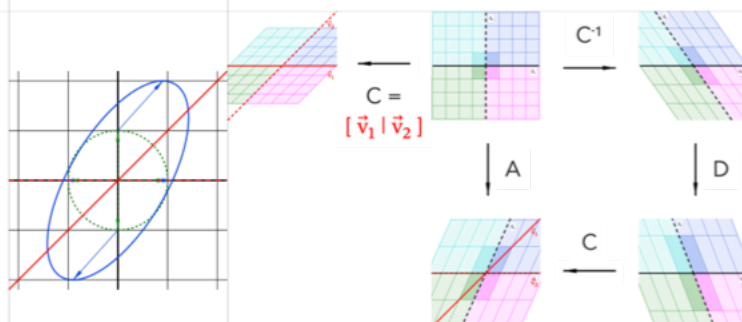
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General non-symmetric

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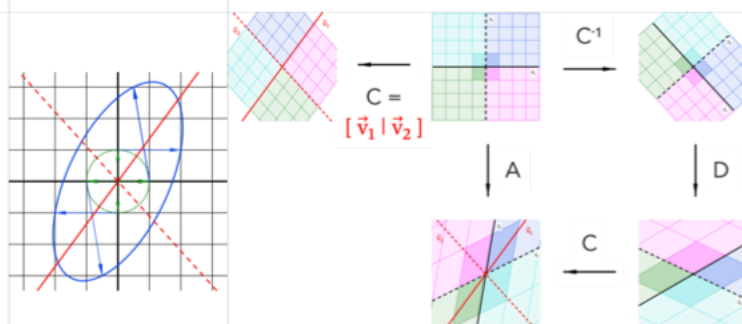
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two $\vec{x} \in \mathbb{R}^2$

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Symmetric positive λ

$$A = Q D Q^T$$

$$\begin{bmatrix} 2 & 0.6 \\ 0.6 & 1.2 \end{bmatrix} =$$

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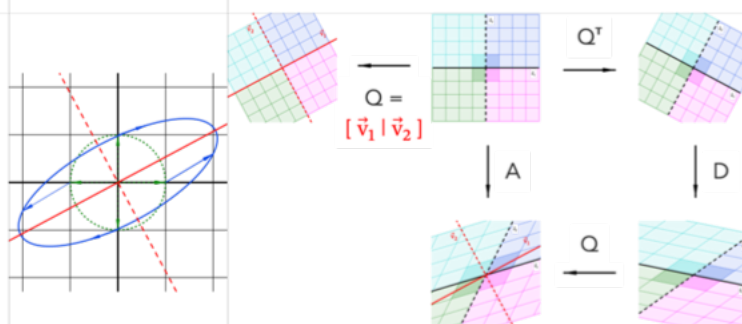
$$\lambda_1 = 2.321 > 1$$

$$0 < \lambda_2 = 0.879 < 1$$

$$\vec{x}_1 \cdot \vec{x}_2 = 0$$

$\vec{v} \rightarrow A \vec{v}$ arrows cross neither eigenline

$$Q = [\vec{x}_1 \mid \vec{x}_2]$$



Symmetric negative λ

$$A = Q D Q^T$$

$$\begin{bmatrix} -2 & -0.6 \\ -0.6 & -1.2 \end{bmatrix} =$$

$$\begin{bmatrix} \approx -0.472 & \approx 0.882 \\ \approx 0.882 & \approx 0.472 \end{bmatrix} \begin{bmatrix} \approx -0.879 & 0 \\ 0 & \approx -2.321 \end{bmatrix} \begin{bmatrix} \approx -0.472 & \approx 0.882 \\ \approx 0.882 & \approx 0.472 \end{bmatrix}$$

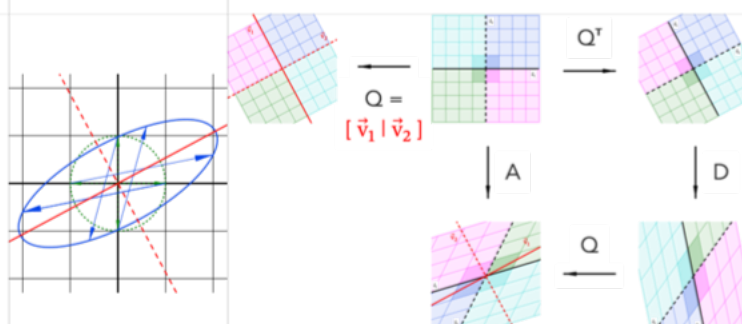
$$-1 < \lambda_1 = -0.879 < 0$$

$$\lambda_2 = -2.321 < -1$$

$$\vec{x}_1 \cdot \vec{x}_2 = 0$$

$\vec{v} \rightarrow A \vec{v}$ arrows cross both eigenlines

$$Q = [\vec{x}_1 \mid \vec{x}_2]$$



Symmetric mixed-sign λ

$$A = Q D Q^T$$

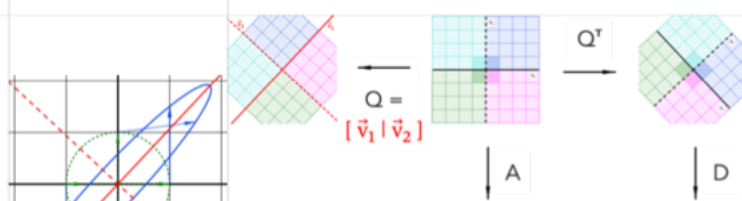
$$\begin{bmatrix} 1 & 1.5 \\ 1.5 & 1.2 \end{bmatrix} =$$

$$\lambda_1 = 2.603 > 1$$

$$-1 < \lambda_2 = -0.403 < 0$$

$$\vec{x}_1 \cdot \vec{x}_2 = 0$$

$\vec{v} \rightarrow A \vec{v}$ arrows cross



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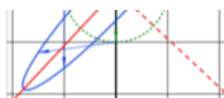
Reflection
also symmetric
 $R = Q D Q^T$

$$\begin{bmatrix} 0.5 & \approx 0.866 \\ \approx 0.866 & -0.5 \end{bmatrix} =$$

$$\begin{bmatrix} \approx 0.866 & -0.5 \\ 0.5 & \approx 0.866 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \approx 0.866 & 0.5 \\ -0.5 & \approx 0.866 \end{bmatrix}$$

the positive- λ eigenline

$$Q = [\vec{x}_1 \mid \vec{x}_2]$$



$$\vec{x}_1 = \text{col}(A) \text{ direction}$$

$$\lambda_1 = 1$$

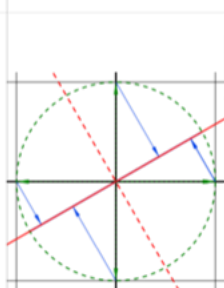
$$\vec{x}_2 = \ell\text{-null}(A) \text{ direction}$$

$$\lambda_2 = 0$$

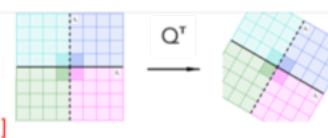
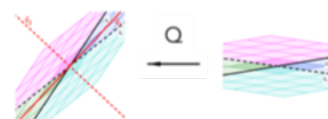
$$\vec{x}_1 \cdot \vec{x}_2 = 0$$

$$\vec{v} \rightarrow A \vec{v} \text{ arrows end on the } \lambda = 1 \text{ eigenline}$$

$$Q = [\vec{x}_1 \mid \vec{x}_2]$$



$$Q = [\vec{v}_1 \mid \vec{v}_2]$$



$$\vec{x}_1 = \text{mirror direction}$$

$$\lambda_1 = 1$$

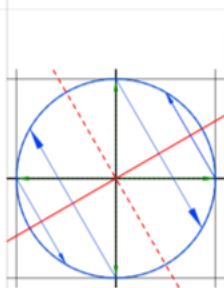
$$\vec{x}_2 = \text{orthogonal direction}$$

$$\lambda_2 = -1$$

$$\vec{x}_1 \cdot \vec{x}_2 = 0$$

$$\vec{v} \rightarrow A \vec{v} \text{ arrows cross the mirror line}$$

$$Q = [\vec{x}_1 \mid \vec{x}_2]$$



$$Q = [\vec{v}_1 \mid \vec{v}_2]$$

