Uniform scaling

$A = C\,D\,C^{-1}$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A\,\vec{x} = \lambda\,\vec{x}$

for every nonzero $\vec{x}$

$\lambda = 1.5 > 1$

$\lambda = A_{11} = A_{22}$

$= D_{11} = D_{22}$

$\downarrow$

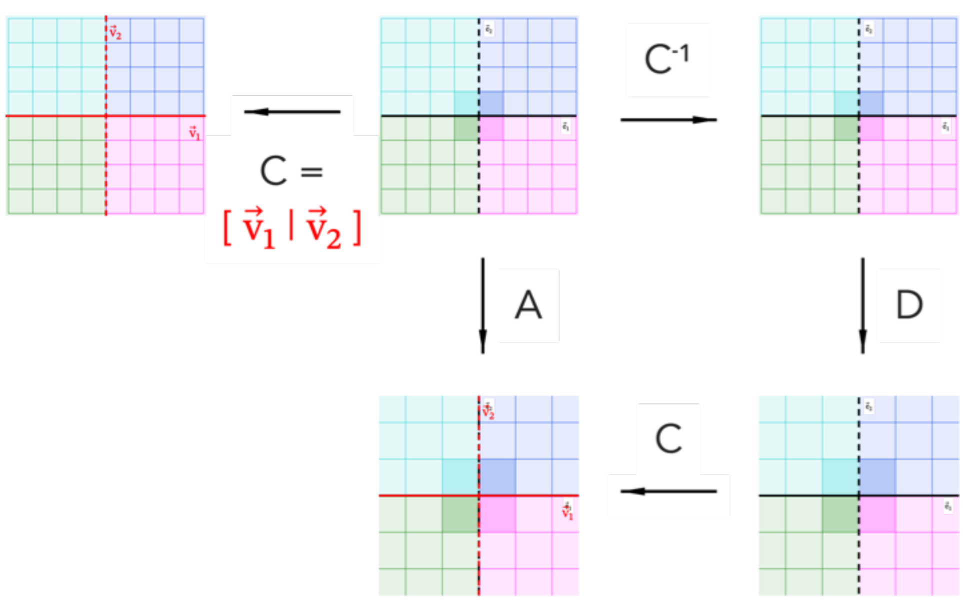
$C\,D = D\,C$

$\downarrow$

any invertible C is valid

shown here:

$C = [\,\vec{e}_1 \mid \vec{e}_2\,] = I$



Upper-triangular

$A = C\,D\,C^{-1}$

$$\begin{bmatrix} 1.1 & 0.9 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & \approx 0.707 \\ 0 & \approx 0.707 \end{bmatrix} \begin{bmatrix} 1.1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & \approx 1.414 \end{bmatrix}$$

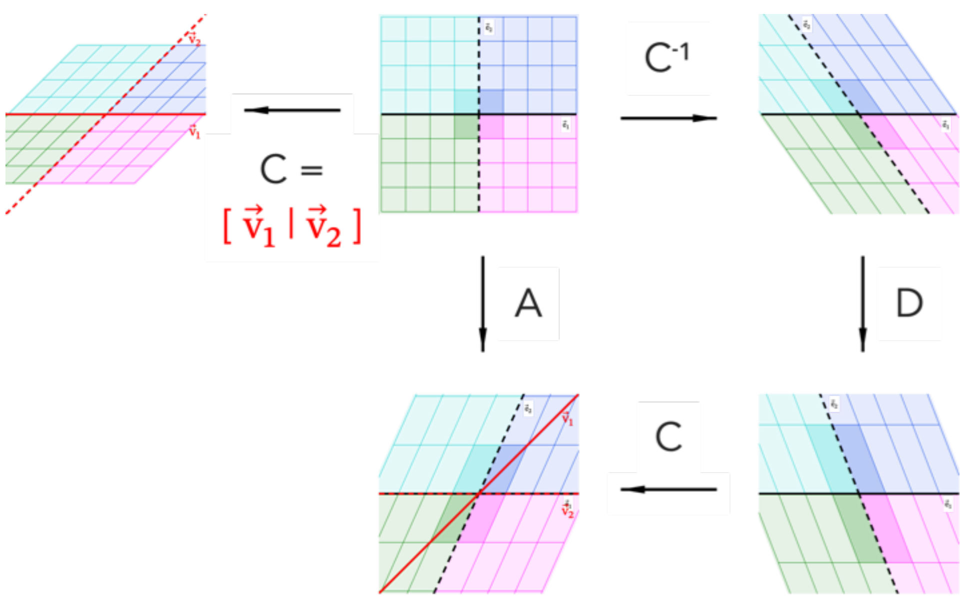
$\lambda_1 = A_{11} = 1.1 > 1$

$\lambda_2 = A_{22} = 2 > 1$

$\vec{x}_1$ is a multiple of $\vec{e}_1$

$\vec{x}_2$ is usually not $\vec{e}_2$

$C = [\ \vec{x}_1 \mid \vec{x}_2\ ]$



General
non-symmetric
$A = C\,D\,C^{-1}$

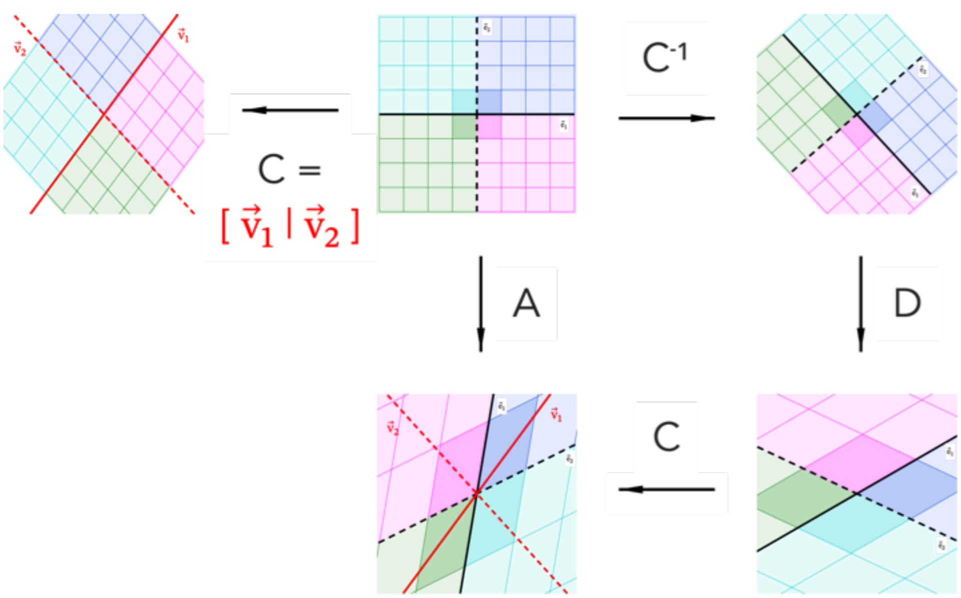
$$\begin{bmatrix} 0.5 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} \approx 0.593 & \approx -0.671 \\ \approx 0.805 & \approx 0.742 \end{bmatrix} \begin{bmatrix} \approx 3.212 & 0 \\ 0 & \approx -1.712 \end{bmatrix} \begin{bmatrix} \approx 0.757 & \approx 0.684 \\ \approx -0.821 & \approx 0.606 \end{bmatrix}$$

$\lambda_1 \approx 3.212 > 1$
$\lambda_2 \approx -1.712 < -1$

two $\vec{x} \in \mathbb{R}^2$

$\vec{x}_1 \cdot \vec{x}_2 \neq 0$

$C = [\, \vec{x}_1 \mid \vec{x}_2 \,]$



Symmetric

positive λ

$A = Q\,D\,Q^T$

$$\begin{bmatrix} 2 & 0.6 \\ 0.6 & 1.2 \end{bmatrix} = \begin{bmatrix} \approx 0.882 & \approx -0.472 \\ \approx 0.472 & \approx 0.882 \end{bmatrix} \begin{bmatrix} \approx 2.321 & 0 \\ 0 & \approx 0.879 \end{bmatrix} \begin{bmatrix} \approx 0.882 & \approx 0.472 \\ \approx -0.472 & \approx 0.882 \end{bmatrix}$$

$\lambda_1 \approx 2.321 > 1$

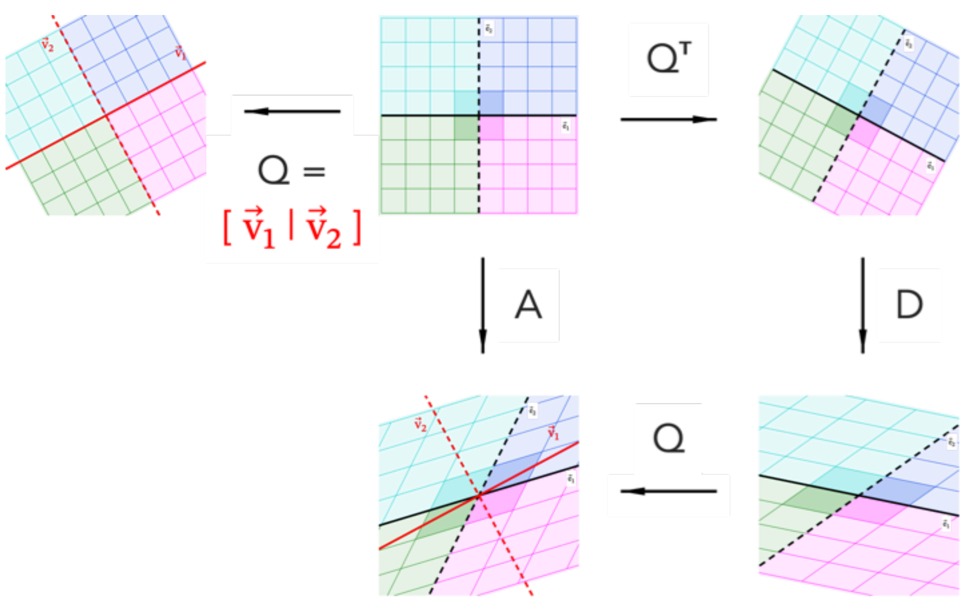
$0 < \lambda_2 \approx 0.879 < 1$

$\vec{x}_1 \cdot \vec{x}_2 = 0$

$\vec{v} \to A\,\vec{v}$ arrows cross

neither eigenline

$Q = [\,\vec{x}_1 \mid \vec{x}_2\,]$



Symmetric

negative λ

$A = Q\,D\,Q^T$

$$\begin{bmatrix} -2 & -0.6 \\ -0.6 & -1.2 \end{bmatrix} = \begin{bmatrix} \approx -0.472 & \approx 0.882 \\ \approx 0.882 & \approx 0.472 \end{bmatrix} \begin{bmatrix} \approx -0.879 & 0 \\ 0 & \approx -2.321 \end{bmatrix} \begin{bmatrix} \approx -0.472 & \approx 0.882 \\ \approx 0.882 & \approx 0.472 \end{bmatrix}$$

$-1 < \lambda_1 \approx -0.879 < 0$

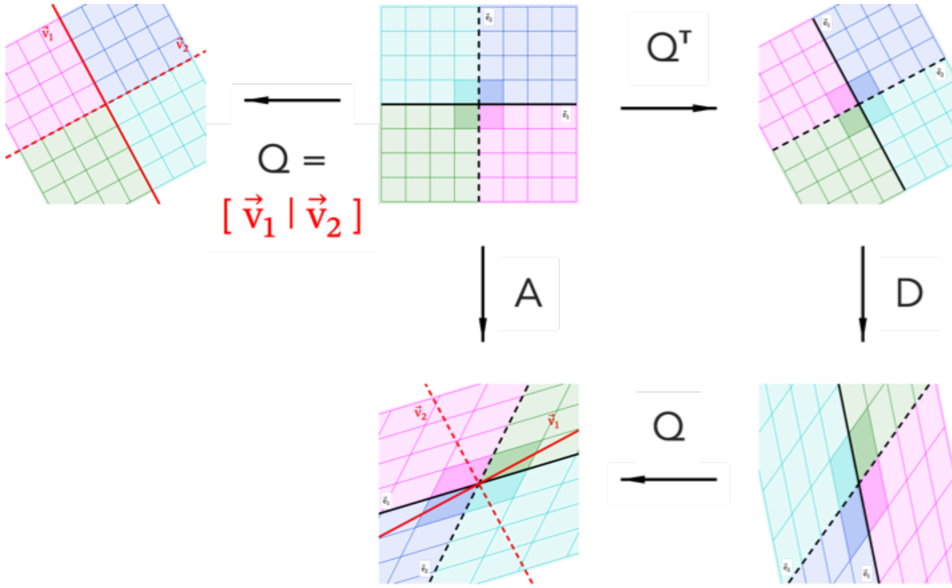
$\lambda_2 \approx -2.321 < -1$

$\vec{x}_1 \cdot \vec{x}_2 = 0$

$\vec{v} \to A\,\vec{v}$ arrows cross

both eigenlines

$Q = [\, \vec{x}_1 \mid \vec{x}_2 \,]$



Symmetric
mixed-sign λ
$A = Q\,D\,Q^T$

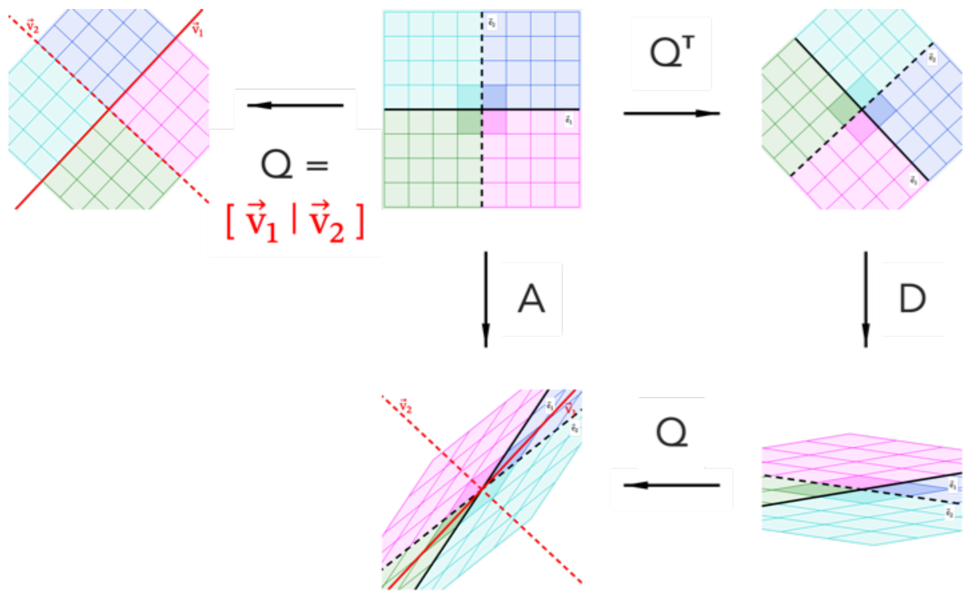
$$\begin{bmatrix} 1 & 1.5 \\ 1.5 & 1.2 \end{bmatrix} = \begin{bmatrix} \approx 0.683 & \approx -0.73 \\ \approx 0.73 & \approx 0.683 \end{bmatrix} \begin{bmatrix} \approx 2.603 & 0 \\ 0 & \approx -0.403 \end{bmatrix} \begin{bmatrix} \approx 0.683 & \approx 0.73 \\ \approx -0.73 & \approx 0.683 \end{bmatrix}$$

$\lambda_1 \approx 2.603 > 1$
$-1 < \lambda_2 \approx -0.403 < 0$

$\vec{x}_1 \cdot \vec{x}_2 = 0$

$\vec{v} \rightarrow A\,\vec{v}$ arrows cross the positive-λ eigenline

$Q = [\ \vec{x}_1 \mid \vec{x}_2\ ]$



Projection
also symmetric
$P = Q\,D\,Q^T$

$$\begin{bmatrix} 0.75 & \approx 0.433 \\ \approx 0.433 & 0.25 \end{bmatrix} = \begin{bmatrix} \approx 0.866 & -0.5 \\ 0.5 & \approx 0.866 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \approx 0.866 & 0.5 \\ -0.5 & \approx 0.866 \end{bmatrix}$$

$\vec{x}_1$ = col(A) direction
$\lambda_1 = 1$

$\vec{x}_2 = \ell$-null(A) direction

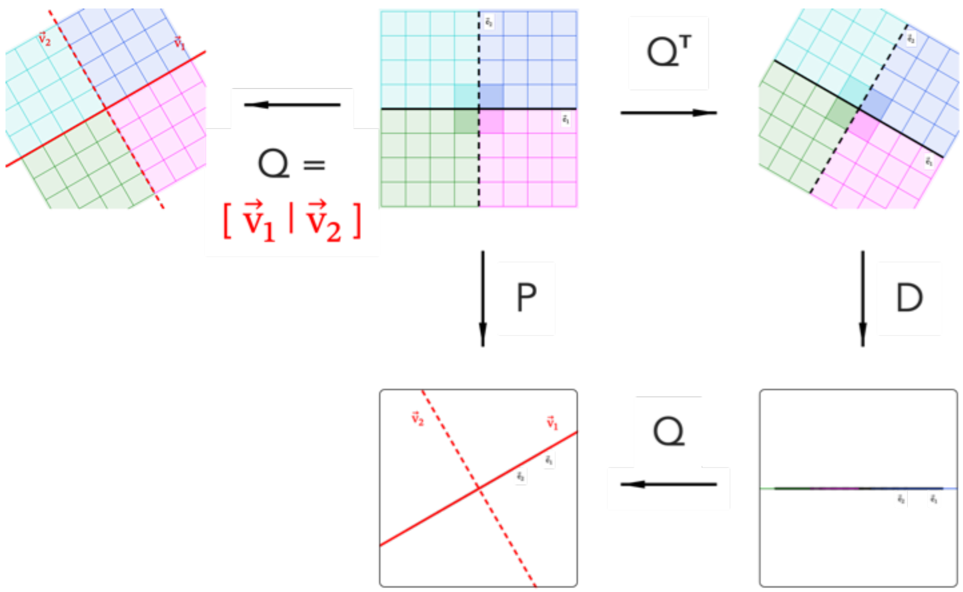
$\lambda_2 = 0$

$\vec{x}_1 \cdot \vec{x}_2 = 0$

$\vec{v} \rightarrow A\,\vec{v}$ arrows end on

the $\lambda = 1$ eigenline

$Q = [\, \vec{x}_1 \mid \vec{x}_2 \,]$



Reflection

also symmetric

$R = Q\,D\,Q^T$

$$\begin{bmatrix} 0.5 & \approx 0.866 \\ \approx 0.866 & -0.5 \end{bmatrix} = \begin{bmatrix} \approx 0.866 & -0.5 \\ 0.5 & \approx 0.866 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \approx 0.866 & 0.5 \\ -0.5 & \approx 0.866 \end{bmatrix}$$

$\vec{x}_1$ = mirror direction

$\lambda_1 = 1$

$\vec{x}_2$ = orthogonal direction

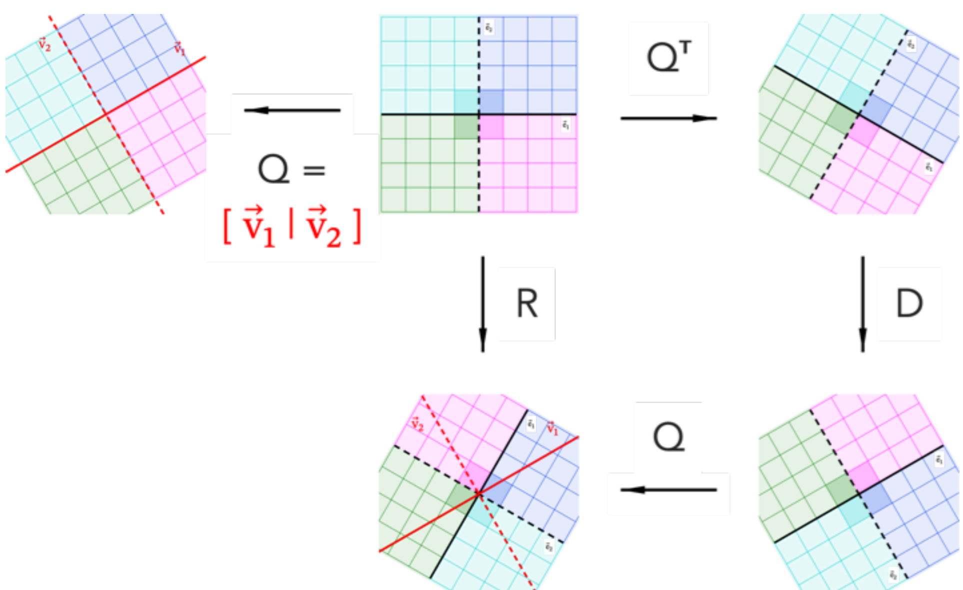
$\lambda_2 = -1$

$\vec{x}_1 \cdot \vec{x}_2 = 0$

$\vec{v} \rightarrow A\,\vec{v}$ arrows cross

the mirror line

$Q = [\,\vec{x}_1 \mid \vec{x}_2\,]$



Diagonalizable matrices with $\lambda \in \mathbb{R}$

Uniform scaling

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$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A\,\vec{x} = \lambda\,\vec{x}$
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↓
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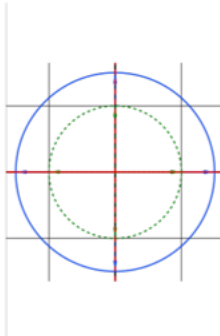

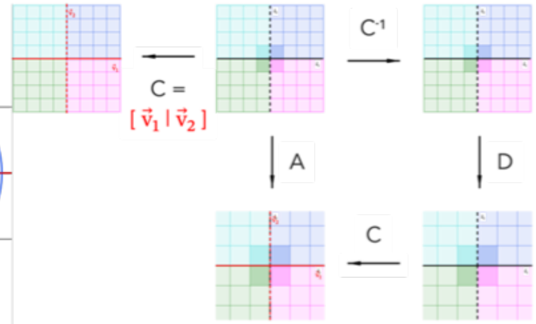

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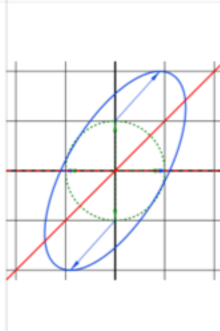

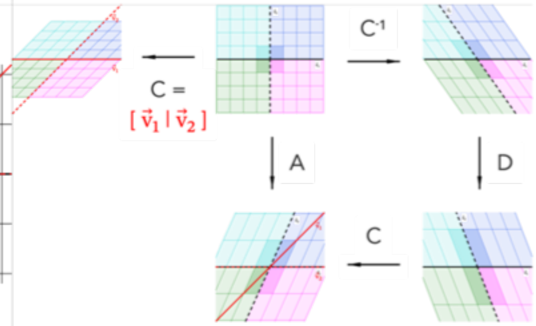

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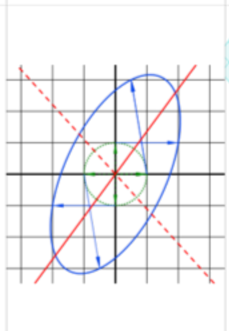
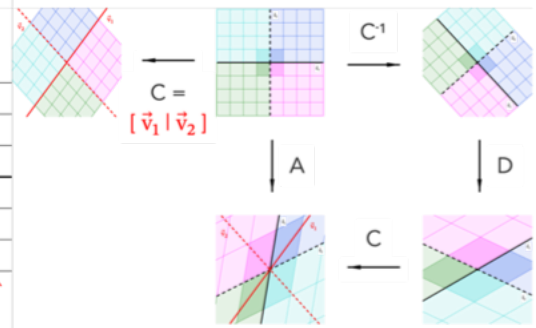
$\vec{x}_1 \cdot \vec{x}_2 \neq 0$

$C = [\,\vec{x}_1 \mid \vec{x}_2\,]$





Symmetric
positive λ

$A = Q\,D\,Q^T$

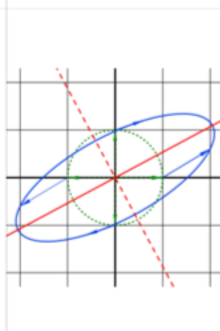
$$\begin{bmatrix} 2 & 0.6 \\ 0.6 & 1.2 \end{bmatrix} = \begin{bmatrix} \approx 0.882 & \approx -0.472 \\ \approx 0.472 & \approx 0.882 \end{bmatrix}\begin{bmatrix} \approx 2.321 & 0 \\ 0 & \approx 0.879 \end{bmatrix}\begin{bmatrix} \approx 0.882 & \approx 0.472 \\ \approx -0.472 & \approx 0.882 \end{bmatrix}$$

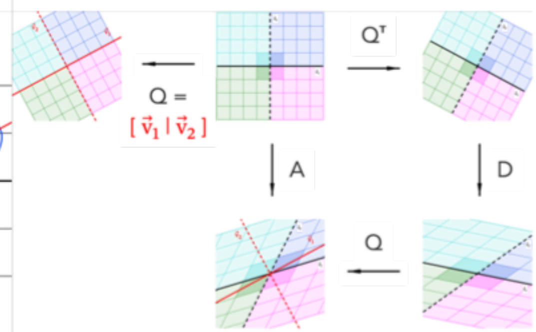
$\lambda_1 \approx 2.321 > 1$
$0 < \lambda_2 \approx 0.879 < 1$

$\vec{x}_1 \cdot \vec{x}_2 = 0$

$\vec{v} \rightarrow A\,\vec{v}$ arrows cross
neither eigenline

$Q = [\,\vec{x}_1 \mid \vec{x}_2\,]$





Symmetric
negative λ

$A = Q\,D\,Q^T$

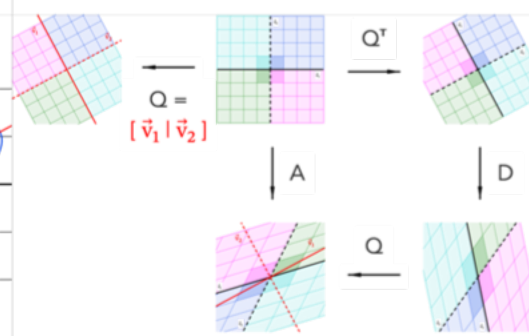
$$\begin{bmatrix} -2 & -0.6 \\ -0.6 & -1.2 \end{bmatrix} = \begin{bmatrix} \approx -0.472 & \approx 0.882 \\ \approx 0.882 & \approx 0.472 \end{bmatrix}\begin{bmatrix} \approx -0.879 & 0 \\ 0 & \approx -2.321 \end{bmatrix}\begin{bmatrix} \approx -0.472 & \approx 0.882 \\ \approx 0.882 & \approx 0.472 \end{bmatrix}$$

$-1 < \lambda_1 \approx -0.879 < 0$
$\lambda_2 \approx -2.321 < -1$

$\vec{x}_1 \cdot \vec{x}_2 = 0$

$\vec{v} \rightarrow A\,\vec{v}$ arrows cross
both eigenlines

$Q = [\,\vec{x}_1 \mid \vec{x}_2\,]$



Symmetric
mixed-sign λ

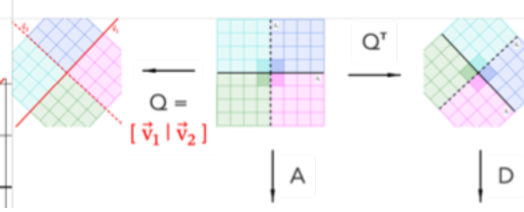
$A = Q\,D\,Q^T$

$$\begin{bmatrix} 1 & 1.5 \\ 1.5 & 1.2 \end{bmatrix} =$$

$\lambda_1 \approx 2.603 > 1$
$-1 < \lambda_2 \approx -0.403 < 0$

$\vec{x}_1 \cdot \vec{x}_2 = 0$

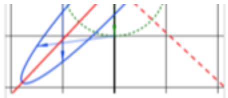
$\vec{v} \rightarrow A\,\vec{v}$ arrows cross

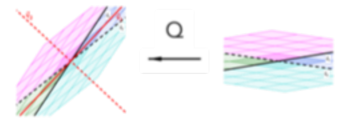


$$\begin{bmatrix} \approx 0.683 & \approx -0.73 \\ \approx 0.73 & \approx 0.683 \end{bmatrix} \begin{bmatrix} \approx 2.603 & 0 \\ 0 & \approx -0.403 \end{bmatrix} \begin{bmatrix} \approx 0.683 & \approx 0.73 \\ \approx -0.73 & \approx 0.683 \end{bmatrix}$$

the positive-λ eigenline

$Q = [\,\vec{x}_1 \mid \vec{x}_2\,]$





Projection
also symmetric
$P = Q\,D\,Q^T$

$$\begin{bmatrix} 0.75 & \approx 0.433 \\ \approx 0.433 & 0.25 \end{bmatrix} =$$

$$\begin{bmatrix} \approx 0.866 & -0.5 \\ 0.5 & \approx 0.866 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \approx 0.866 & 0.5 \\ -0.5 & \approx 0.866 \end{bmatrix}$$

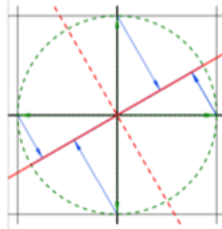
$\vec{x}_1$ = col(A) direction
$\lambda_1 = 1$

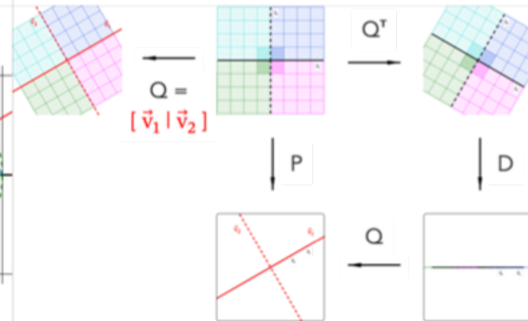
$\vec{x}_2$ = ℓ-null(A) direction
$\lambda_2 = 0$

$\vec{x}_1 \cdot \vec{x}_2 = 0$

$\vec{v} \to A\,\vec{v}$ arrows end on the $\lambda = 1$ eigenline

$Q = [\,\vec{x}_1 \mid \vec{x}_2\,]$





Reflection
also symmetric
$R = Q\,D\,Q^T$

$$\begin{bmatrix} 0.5 & \approx 0.866 \\ \approx 0.866 & -0.5 \end{bmatrix} =$$

$$\begin{bmatrix} \approx 0.866 & -0.5 \\ 0.5 & \approx 0.866 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \approx 0.866 & 0.5 \\ -0.5 & \approx 0.866 \end{bmatrix}$$

$\vec{x}_1$ = mirror direction
$\lambda_1 = 1$

$\vec{x}_2$ = orthogonal direction
$\lambda_2 = -1$

$\vec{x}_1 \cdot \vec{x}_2 = 0$

$\vec{v} \to A\,\vec{v}$ arrows cross the mirror line

$Q = [\,\vec{x}_1 \mid \vec{x}_2\,]$