Definition

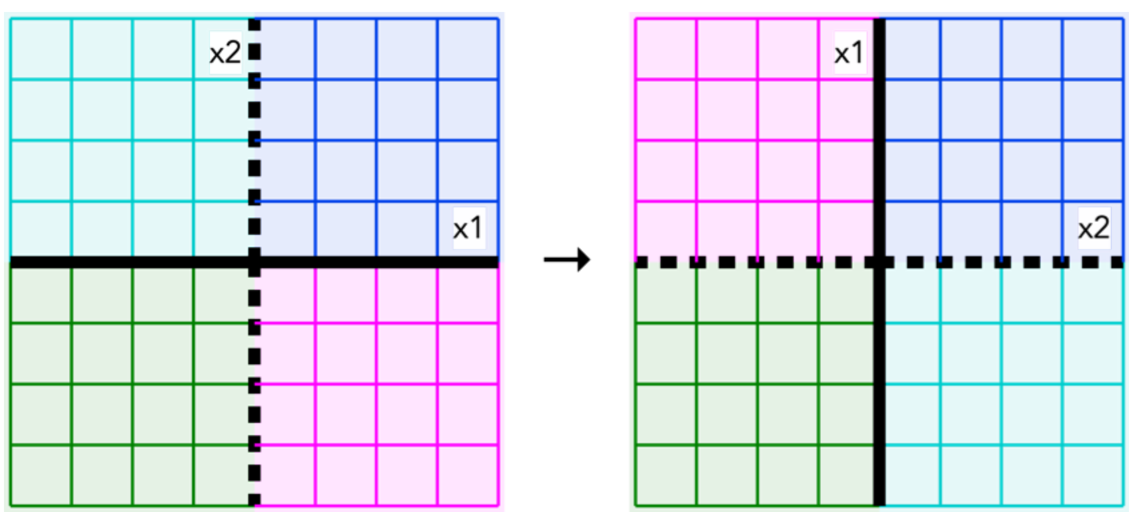
Determinant is a unique scalar associated with a linear transformation
and computed from any square matrix M that represents it
and is defined as follows:

- | I | = 1 (I = identity matrix)
- Swapping rows of M reverses sign of |M|
- Scaling one row of M by a constant scales |M| by the same constant
- Adding multiple of one row of M to another row leaves |M| unchanged

Any full-rank matrix can be obtained from the identity by a finite sequence of these operations
Therefore the determinant is well-defined for any invertible matrix
as the product of the corresponding elementary steps
If a matrix is not full-rank, its determinant is zero, as shown later

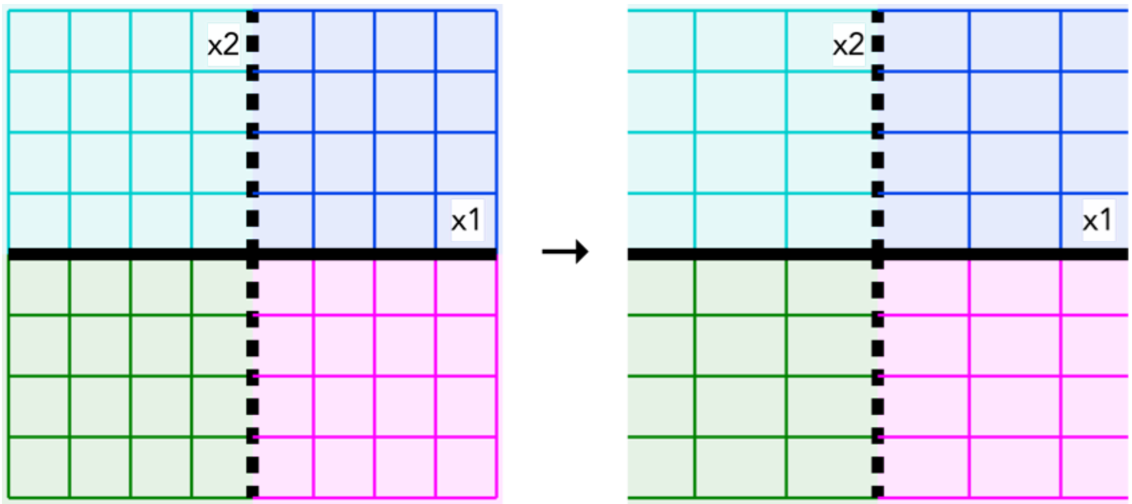
Images below show transformation by elementary matrices:

Row swap: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



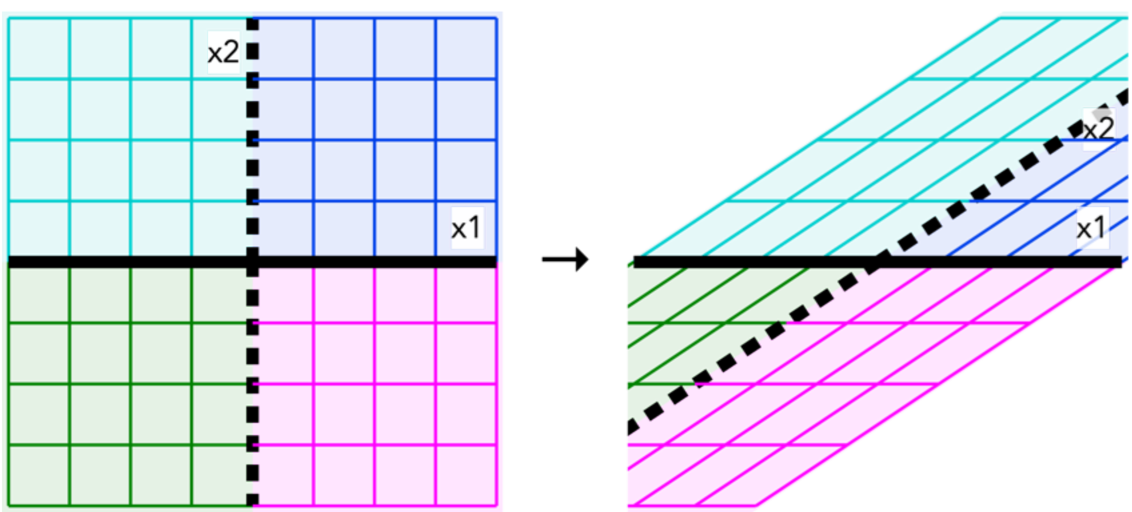
volume is unchanged, direction is reversed

Row scaling: $\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}$



volume is scaled by 1.5

Row replacement or shear: $\begin{bmatrix} 1 & 1.5 \\ 0 & 1 \end{bmatrix}$



volume is unchanged

The images illustrate geometric effect of
elementary matrix transformations on volume (area in 2D)

Determinant was defined algebraically to represent
signed scaling of n-dimensional volume

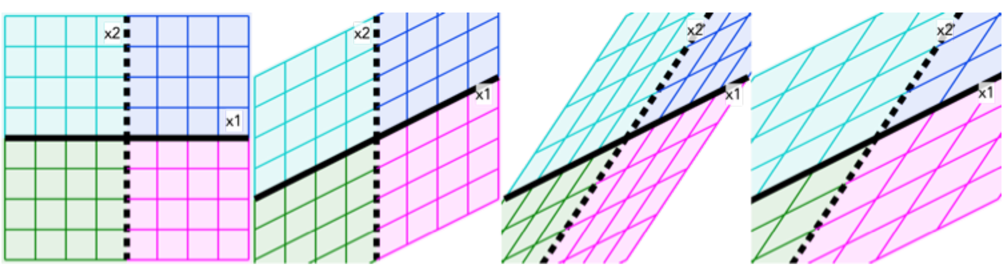
Consider the following:
Any parallelogram can be obtained from unit square by applying a series of reflections, scalings, and shears
This is the geometric equivalent of writing M as product of elementary matrices (or their inverses)

At each elementary step, the volume transforms in one of the following ways

- change sign: reflection
- scale by a constant: scaling along one row
- remain unchanged: shear along one axis

Below is an example of invertible matrix $M = \begin{bmatrix} 2 & 1 \\ 1 & 1.5 \end{bmatrix} =$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$



Same logic applies to 3 or n dimensions

One more algebraic property (definition level):
Fix all rows of M except one

If a row is written as a sum of two row vectors,
the determinant splits accordingly, as written below:

$M = [\ m_1 \mid m_3\ ]$

$N = [\ m_2 \mid m_3\ ]$

$O = [\ m_1 + m_2 \mid m_3\ ]$

$|\ O\ | = |\ M\ | + |\ N\ |$

Determinant of elementary matrices

Recall from 'Echelon & reduced echelon' tutorial:

- Elementary matrix of row swap is obtained from I by swapping rows i & j
- Elementary matrix of row scaling is obtained from I by scaling row i by scalar s
- Elementary matrix of row replacement is obtained from I by adding a multiple of row i to row j

(I = identity matrix)

Determinant rules applied to elementary matrices:

- | I | = 1 (I = identity matrix)
- Swapping rows of M reverses sign of |M|
- Scaling one row of M by a constant scales |M| by the same constant
- Adding multiple of one row of M to another row leaves |M| unchanged

⇩

- | E of row swap | = -1
- | E of row scaling | = s
- | E of row replacement | = 1

Determinant of non-full-rank M

If M is a non-full-rank square matrix,
at least one row of M can be written as linear combination of others

Since row replacement operations do not change determinant,
we can use them to obtain a new matrix N such that

- $|N| = |M|$
- N has at least one all-0 row, which we denote as n(i)

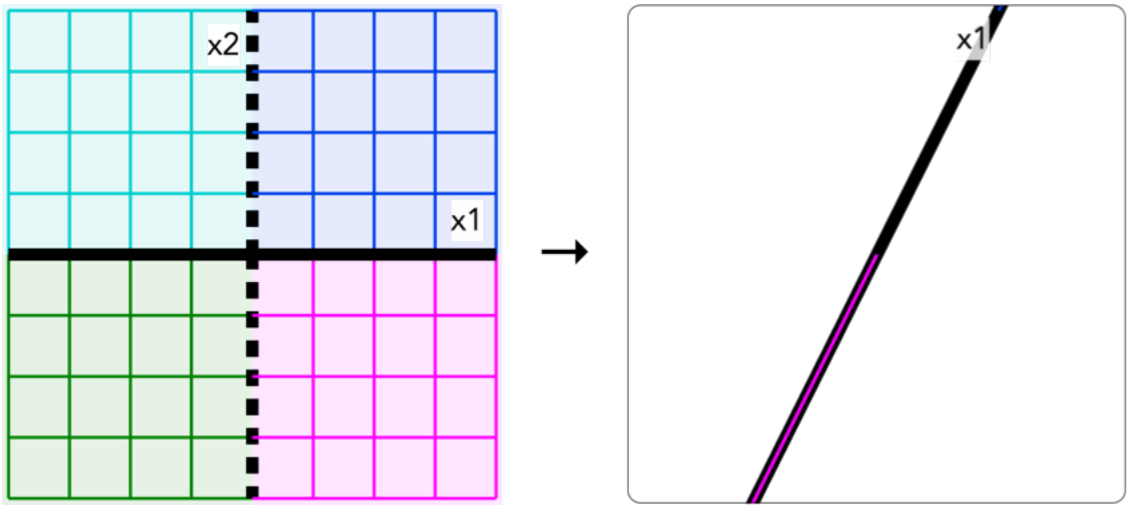
Recall: multiplying one row of N by a constant s
multiplies the determinant $|N|$ by s

If N contains an all-0 row, then scaling that row by any s
does not change N (the row remains all zeros)
So $|N|$ must stay the same

Therefore $|N| = s|N|$ for all s
The only number satisfying this is $|N| = 0$

The image below shows the linear transformation by N:

$$\text{non-full-rank matrix } M = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$



M multiplies volume of the grid by | M | = 0

| AB | = | A |×| B |

Recall determinant definition:

- | I | = 1 (I = identity matrix)
- Swapping rows of M reverses sign of |M|
- Scaling one row of M by a constant scales |M| by the same constant
- Adding multiple of one row of M to another row leaves |M| unchanged

Performing an elementary row operation is equivalent to left-multiplication by an elementary matrix E

Recall

- | E of row swap | = -1
- | E of row scaling | = s
- | E of row replacement | = 1

⇩

$|EA| = |E||A|$ for any full-rank square matrix A

If row reduction turns A into I, then there exist elementary matrices $E1...Ej$ such that

$$Ej...E1\ A = I$$

Therefore $A = F1...Fj$ where $Fi = (Ei)^{-1}$ (inverse elementary matrices)

Similarly, for B there exist elementary matrices $G1...Gk$ such that

$$Gk...G1\ B = I$$

Therefore $B = H1...Hk$ where $Hi = (Gi)^{-1}$

Any full-rank matrix A can be presented as product of inverse elementary matrices:

$$A = F1...Fj$$

$$\Downarrow$$

$$|A| = |F1...Fj| = |F1|...|Fj|$$

Any full-rank matrix B can be presented as product of inverse elementary matrices:

$$B = H1...Hk$$

$$\Downarrow$$

$$|B| = |H1...Hk| = |H1|...|Hk|$$

From above follows:

$$|AB| = |A||B|$$

Non-full-rank case

If A is not full rank, then $|A| = 0$

The product AB is also not full rank, so $|AB| = 0$

Therefore $|AB| = |A||B| = 0$

The same argument applies if B is not full rank

Geometric interpretation (illustration)
B scales volume by |B|, then A scales it by |A| again
Total scaling under AB is |A||B|

Determinant of a triangular matrix and
determinant computation

Consider a square and upper triangular matrix T of size n=3 or any n:

$$\begin{bmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{bmatrix}$$

We can derive another upper triangular matrix U =

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

in such way that $(\text{row } i \text{ of } U) = \dfrac{(\text{row } i \text{ of } T)}{t_{ii}}$

$$\text{In this case, } |T| = |U| \times \prod_{i=1}^{n} (t_{ii})$$

U has a pivot in every column

⇓

U can be row reduced to identity matrix with | I | = 1
using row replacements and without need for row swaps

⇓

$$|U| = 1 \rightarrow |T| = \prod_{i=1}^{n} (t_{ii})$$

This tells us how to compute determinant in practice:
Row-reduce M to an upper triangular matrix T,
track any row swaps and row scalings,
then multiply the diagonal entries of T

Specifically:

- each row swap multiplies determinant by -1
- scaling a row by s multiplies determinant by s
- row replacements do not change determinant

After all steps, if the final triangular matrix has diagonal entries t_{11}, t_{22}, …, t_{nn}, then
|M| = (sign from swaps) × (product of all scaling factors) × ($t_{11}t_{22}\dots t_{nn}$)
If M is non-full-rank, T will have a zero on the diagonal, producing |M| = 0

Determinant of M transposed

First, will examine $|E^T|$ for all elementary matrices

$$\text{E of row swap} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

matrix is symmetric: $E^T = E$

$$\text{E of row scaling} \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E^T = \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

matrix is symmetric: $E^T = E$

$$\text{E of row replacement} \begin{bmatrix} 1 & 0 & 0 & 0 \\ s & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E^T = \begin{bmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

E^T is simply another row replacement matrix

From above, we can conclude that for any E, $|E| = |E^T|$

If M is full-rank, it can be presented as product of inverses of elementary matrices (inverse of an elementary matrix is an elementary matrix)

Suppose M = E1...Ek

$$\Downarrow$$

$|M| = |E1|...|Ek|$

$$\Downarrow$$

$|M^T| = |Ek^T|...|E1^T| = |Ek|...|E1| = |E1|...|Ek| = |M|$

If M is not full-rank, then $\text{rank}(M^T) = \text{rank}(M)$, so both determinants are zero

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Determinant of orthonormal matrix

Consider a square full-rank matrix Q of size n=3 or any other n:

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

Q consists of rows q1...qn that satisfy the following:

- $qi \cdot qi = 1$ for any i
- $qi \cdot qj = 0$ for any $i \neq j$

so that QQ^T becomes

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} q_{11} & q_{21} & q_{31} \\ q_{12} & q_{22} & q_{32} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} =$$

$$\begin{bmatrix} q1 \cdot q1 & q1 \cdot q2 & q1 \cdot q3 \\ q2 \cdot q1 & q2 \cdot q2 & q2 \cdot q3 \\ q3 \cdot q1 & q3 \cdot q2 & q3 \cdot q3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

An orthonormal square matrix satisfies:

$QQ^T = I$ & $Q^T Q = I$ (can be shown in similar fashion)

Orthonormal matrices represent rotations and reflections
(later on, we will show how they preserve lengths and angles)

If $QQ^T = I$, then $|Q||Q^T| = |I| = 1$

$$\Downarrow$$

$|Q| = \{ -1, 1 \}$

Illustration of permutation method

Permutation method

(not used in practice for computation, mainly of historical significance)

Consider an n×n (n=3) matrix M = $\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$

There are n! = 3! = 6 possible ways to reorder column indices { 1, 2, 3 }, called permutations σ

Each permutation has a sign:

positive if there is an even number of inversions in the sequence

negative if there is an odd number of such inversions

List of permutations of columns of M:

- σ_1 = { 1, 2, 3 }

  inversions: none; sign: positive
- σ_2 = { 1, 3, 2 }

  inversions: 3→2; sign: negative
- σ_3 = { 2, 1, 3 }

  inversions: 2→1; sign: negative
- σ_4 = { 2, 3, 1 }

  inversions: 2→1, 3→1; sign: positive
- σ_5 = { 3, 1, 2 }

  inversions: 3→1, 3→2; sign: positive
- σ_6 = { 3, 2, 1 }

  inversions: 3→2, 3→1, 2→1; sign: negative

• |M| is a sum of n!=6 terms

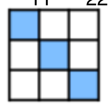
• Each σ contributes one term & provides the sign of the term

- Each term has n factors
- Each factor is an entry of M: m1, σ(1), m2, σ(2), m3, σ(3)

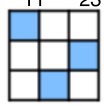
or, equivalently,

Each σ selects exactly one entry from each row by choosing column σ(i) in row i as illustrated below:

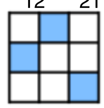
- $+m_{11}m_{22}m_{33}$



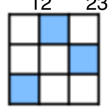
- $-m_{11}m_{23}m_{32}$



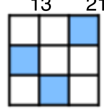
- $-m_{12}m_{21}m_{33}$



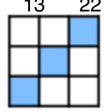
- $+m_{12}m_{23}m_{31}$



- $+m_{13}m_{21}m_{32}$



- $-m_{13}m_{22}m_{31}$



This algorithm generalizes to n×n matrices

How this ties with the algebraic definition:

① Swapping rows of M reverses sign of |M|:
swapping rows will reverse the sign of every term

② Scaling one row of M by a constant scales |M| by the same constant:
scaling one row will scale every term by the same scalar

③ Adding multiple of one row of M to another row leaves |M| unchanged:
adding a multiple of another row produces cancellation
because terms come in pairs with opposite signs
when two rows “compete” for the same column choice

Illustration of cofactor expansion method

Cofactor expansion method
(not used in practice for computation, mainly of historical significance)

① Consider 2×2 matrix $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

$$| C | = c_{11}c_{22} - c_{12}c_{21}$$

(row reduction is one way to derive this formula)

② Consider 3×3 matrix $M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$

| M | can be computed by expansion along any row or column:
will choose row 1 for this illustration

(for computational efficiency, you can use any row or column with the most 0 entries)

$$m_{11} \times \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} - m_{12} \times \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} + m_{13} \times \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix}$$

- Note alternating signs of each term
- Each term uses a minor $|M_{1j}|$; the cofactor is $C_{1j} = (-1)^{(1+j)} |M_{1j}|$
- More generally: $|M| = \sum_{j=1}^{n} (m_{1j} C_{1j})$

The 3×3 example demonstrates the recursion:
each determinant is reduced to several smaller ones
For larger matrices this quickly becomes impractical,
so determinant computation is done using row reduction

⚠️ Personal digression!
Not required for Linear Algebra fluency,
included only to close logical gaps
Read at your own peril: you are warned 😂

How this ties with the algebraic definition:

① Swapping rows of M reverses sign of |M|:
In cofactor expansion, the sign is tied to the row index
Swapping rows moves entries into positions with opposite signs, so every term changes sign

② Scaling one row of M by a constant scales |M| by the same constant:
We can expand along any row or column, thus,
scaling any row will scale every term by the same factor

③ Adding multiple of one row of M to another row leaves |M| unchanged:

- Will calculate | M | by expanding along row 1 (r1):

$$| M | = \sum_{j=1}^{n} \left(m_{1j} \, C_{1j} \right)$$

- Will create a new matrix N identical to M except r1(N) = r1(M) + s r2(M)
- Will calculate | N | by expanding along row 1:

$$| N | = \sum_{j=1}^{n} \left(\left(m_{1j} + s \, m_{2j} \right) C_{1j} \right) =$$

$$\sum_{j=1}^{n} \left(m_{1j} \, C_{1j} \right) + s \sum_{j=1}^{n} \left(m_{2j} C_{1j} \right)$$

- The first sum is exactly |M|
- The second sum equals | O |, where O is M with row 1 replaced by row 2
In O, rows 1 and 2 are identical

- From (1): swapping rows 1 and 2 negates the determinant, so |swap(O)| = −|O|
But swap(O) = O (matrix is unchanged), so | O | = −| O | ⇒ | O | = 0

Therefore:
| N | = | M |

Summary

- Determinants are introduced as scalars associated with square matrices, defined by how they change under elementary row operations

- The determinant is defined by the following properties:
  - change sign under row swaps,
  - scale under row scaling,
  - remain unchanged under row replacement,
  - satisfy $|I| = 1$

- From these rules follow the key consequences:
  - matrices with dependent rows have determinant 0,
  - full-rank matrices have nonzero determinant,
  - determinant can be computed via row reduction

- Geometric view:
  - interpret the determinant as signed volume
  - view any parallelepiped as the image of the unit cube under a sequence of reflections, scalings, and shears, with the determinant equal to the product of the volume changes at each step

- Explicit formulas (2×2, cofactor expansion, permutation formulas) illustrate structure and existence, but are not used for practical computation.