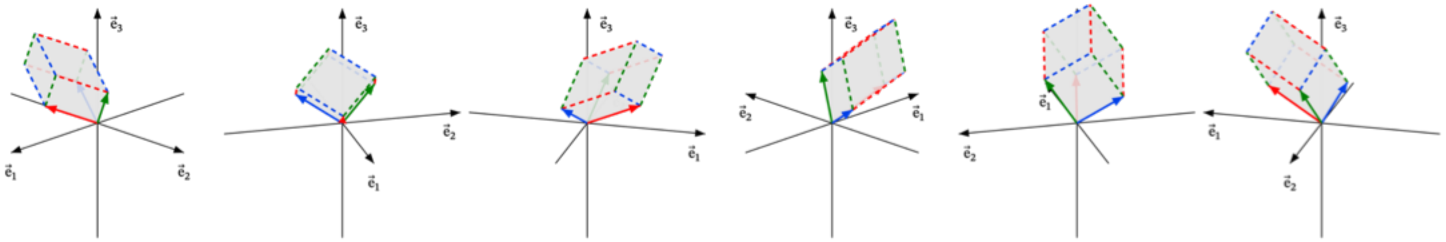
Sign visualization in 3D $\left(\det(A) > 0\right)$

Invertible 3×3 matrix $A = [\, \vec{a}_1 \mid \vec{a}_2 \mid \vec{a}_3 \,] = \left[\begin{array}{c|c|c} 4 & 1 & -1 \\ \hline -1 & 2 & -3 \\ \hline 2 & 3 & 2 \end{array}\right]$ $\det(A) = 55$

det(A) is the signed volume enclosed by the parallelepiped:



Determination of det(A) sign:

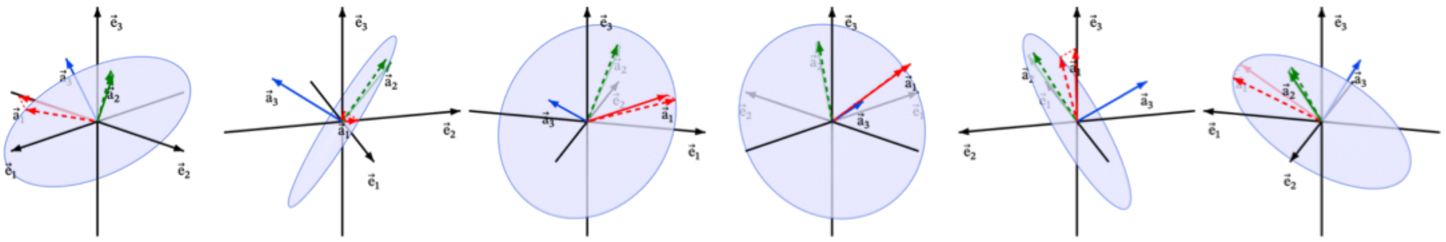
①

- project onto $\vec{a}_3\perp$
- view from the tip of $\vec{a}_3$
- counterclockwise order of projections

matches the cyclic column order $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \ldots$

↓

$\det(A) > 0$



②

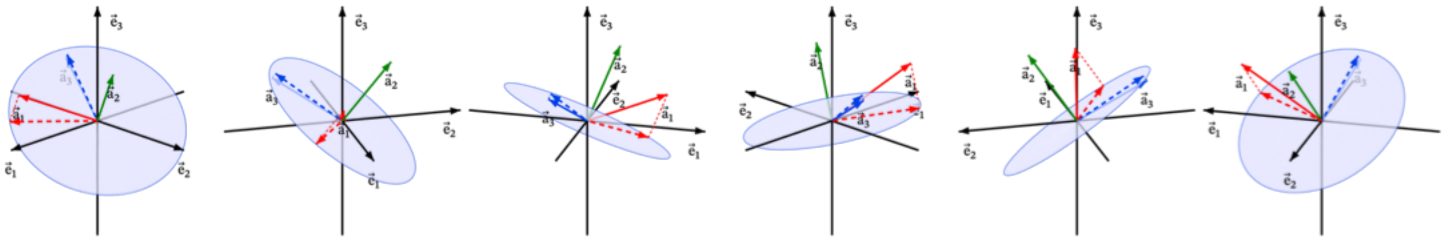
- project onto $\vec{a}_2\perp$

- view from the tip of $\vec{a}_2$
- counterclockwise order of projections

matches the cyclic column order $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \ldots$

↓

det(A) > 0



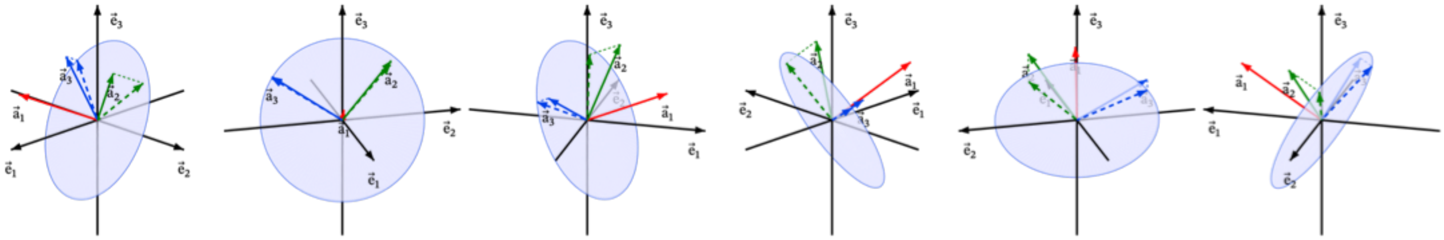
③

- project onto $\vec{a}_1\perp$
- view from the tip of $\vec{a}_1$
- counterclockwise order of projections

matches the cyclic column order $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \ldots$

↓

det(A) > 0



Ⓐ

How is the projection plane defined?

For each method, choose one column vector as the viewing vector

Then project the two remaining column vectors onto the plane orthogonal to it:

- method ①: project $\vec{a}_1$ and $\vec{a}_2$ onto $\vec{a}_3\perp$
- method ②: project $\vec{a}_3$ and $\vec{a}_1$ onto $\vec{a}_2\perp$
- method ③: project $\vec{a}_2$ and $\vec{a}_3$ onto $\vec{a}_1\perp$

Ⓑ
How is the sign read visually?
Look from the tip of the viewing vector toward the origin
If the projected arrows appear in the cyclic column order
$\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_1$, ... then $\det(A) > 0$
If the order is reversed, then $\det(A) < 0$

Ⓒ
Why do methods ② and ③ produce results identical to method ①?

Method ②:
Suppose $B = [\, \vec{b}_1 \mid \vec{b}_2 \mid \vec{b}_3 \,] = [\, \vec{a}_3 \mid \vec{a}_1 \mid \vec{a}_2 \,]$
B is obtained from A by 2 column exchanges
↓
$\det(B) = \det(A)$

Applying method ① to B means projecting onto $\vec{b}_3\perp$
$\vec{b}_3 = \vec{a}_2$
↓
method ① for B is identical to method ② for A

Method ③:
Suppose $C = [\, \vec{c}_1 \mid \vec{c}_2 \mid \vec{c}_3 \,] = [\, \vec{a}_2 \mid \vec{a}_3 \mid \vec{a}_1 \,]$
C is obtained from A by 2 column exchanges
↓
$\det(C) = \det(A)$

Applying method ① to C means projecting onto $\vec{c}_3\perp$
$\vec{c}_3 = \vec{a}_1$
↓
method ① for C is identical to method ③ for A

This works directly because $n = 3$ is odd:
cyclic column shifts preserve determinant sign in odd dimensions

and reverse determinant sign in even dimensions

Ⓓ

What if det(A) = 0?

after projection onto $\vec{a}_3\perp$, the two projected vectors are linearly dependent

↓

no counterclockwise order is defined

Ⓔ

The visualization reduces the determinant sign problem
to reading orientation in the projection plane:

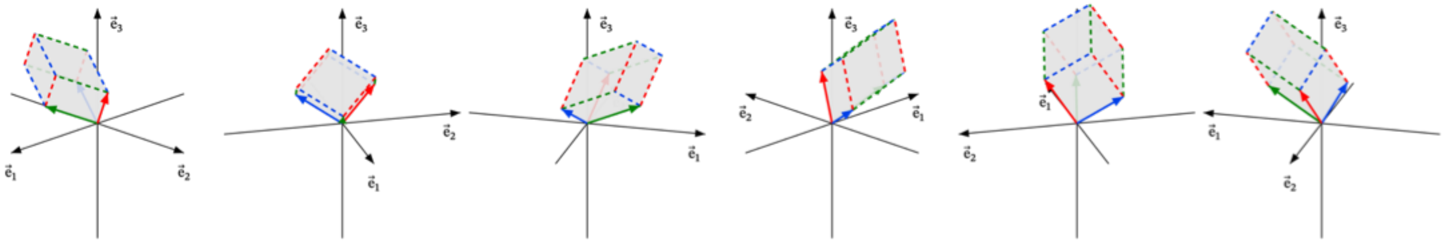
- choose one vector as the viewing vector
- project the other two vectors onto the orthogonal plane
- look from the tip of the viewing vector toward the origin
- compare the visible counterclockwise order with the cyclic column order

If the projected arrows appear in cyclic order, $\det(A) > 0$
If the order is reversed, $\det(A) < 0$
If the projected arrows are linearly dependent, $\det(A) = 0$

Sign visualization in 3D $\left(\det(A) < 0\right)$

Invertible 3×3 matrix $A = [\, \vec{a}_1 \mid \vec{a}_2 \mid \vec{a}_3 \,] = \left[\begin{array}{c|c|c} 1 & 4 & -1 \\ \hline 2 & -1 & -3 \\ \hline 3 & 2 & 2 \end{array}\right]$ $\det(A) = -55$

det(A) is the signed volume enclosed by the parallelepiped:



Determination of det(A) sign:

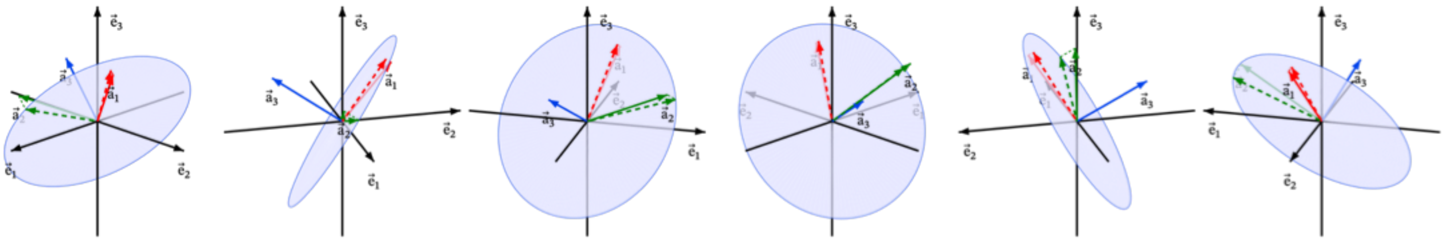
①

- project onto $\vec{a}_3\perp$
- view from the tip of $\vec{a}_3$
- counterclockwise order of projections

does not match the cyclic column order $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_1$, ...

↓

det(A) < 0



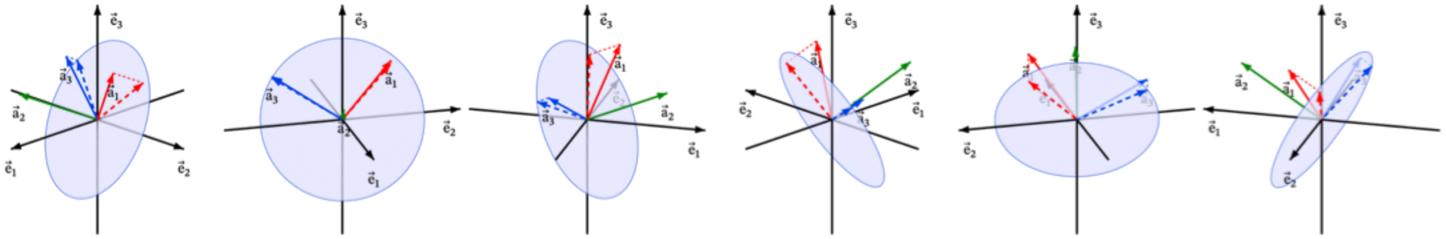
②

- project onto $\vec{a}_2\perp$
- view from the tip of $\vec{a}_2$
- counterclockwise order of projections

does not match the cyclic column order $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_1$, ...

↓

det(A) < 0



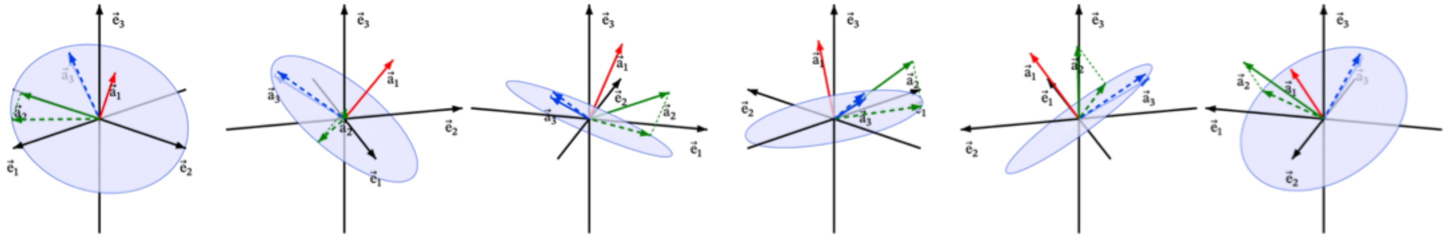
③

- project onto $\vec{a}_1\perp$
- view from the tip of $\vec{a}_1$
- counterclockwise order of projections

does not match the cyclic column order $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \ldots$

↓

$\det(A) < 0$



Ⓐ

How is the projection plane defined?

For each method, choose one column vector as the viewing vector

Then project the two remaining column vectors onto the plane orthogonal to it:

- method ①: project $\vec{a}_1$ and $\vec{a}_2$ onto $\vec{a}_3\perp$
- method ②: project $\vec{a}_3$ and $\vec{a}_1$ onto $\vec{a}_2\perp$
- method ③: project $\vec{a}_2$ and $\vec{a}_3$ onto $\vec{a}_1\perp$

Ⓑ

How is the sign read visually?

Look from the tip of the viewing vector toward the origin

If the projected arrows appear in the cyclic column order

$\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \ldots$ then $\det(A) > 0$

If the order is reversed, then $\det(A) < 0$

Ⓒ

Why do methods ② and ③ produce results identical to method ①?

Method ②:
Suppose $B = [\ \vec{b}_1 \mid \vec{b}_2 \mid \vec{b}_3\ ] = [\ \vec{a}_3 \mid \vec{a}_1 \mid \vec{a}_2\ ]$
B is obtained from A by 2 column exchanges
↓
det(B) = det(A)

Applying method ① to B means projecting onto $\vec{b}_3\perp$
$\vec{b}_3 = \vec{a}_2$
↓
method ① for B is identical to method ② for A

Method ③:
Suppose $C = [\ \vec{c}_1 \mid \vec{c}_2 \mid \vec{c}_3\ ] = [\ \vec{a}_2 \mid \vec{a}_3 \mid \vec{a}_1\ ]$
C is obtained from A by 2 column exchanges
↓
det(C) = det(A)

Applying method ① to C means projecting onto $\vec{c}_3\perp$
$\vec{c}_3 = \vec{a}_1$
↓
method ① for C is identical to method ③ for A

This works directly because $n = 3$ is odd:
cyclic column shifts preserve determinant sign in odd dimensions
and reverse determinant sign in even dimensions

Ⓓ
What if det(A) = 0?
after projection onto $\vec{a}_3\perp$, the two projected vectors are linearly dependent
↓
no counterclockwise order is defined

Ⓔ
The visualization reduces the determinant sign problem

to reading orientation in the projection plane:

- choose one vector as the viewing vector
- project the other two vectors onto the orthogonal plane
- look from the tip of the viewing vector toward the origin
- compare the visible counterclockwise order with the cyclic column order

If the projected arrows appear in cyclic order, det(A) > 0
If the order is reversed, det(A) < 0
If the projected arrows are linearly dependent, det(A) = 0

Sign visualization in higher dimensions
Determinant sign as a recursive projection rule

4D version:
For $A = [\ \vec{a}_1 \mid \vec{a}_2 \mid \vec{a}_3 \mid \vec{a}_4\ ]$:

①
Start with $\vec{a}_4$ as the viewing vector:
ⓐ Project $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ onto $\vec{a}_4\perp$
this reduces the problem to a 3D orientation
ⓑ choose a projected vector (say $\vec{a}_3$) and project the remaining two onto $\vec{a}_3\perp$
we now have a 2D orientation problem where
the final 2D counterclockwise order matches det(A)

②
Start with $\vec{a}_3$ as the viewing vector:

To place $\vec{a}_3$ last, one column exchange is needed
This introduces a sign flip,
so after the same projection steps, the result is the opposite sign of det(A)

③
Start with $\vec{a}_2$:
Placing $\vec{a}_2$ last requires two exchanges, which preserve the sign
After projecting onto $\vec{a}_2\perp$ and following the same steps,
the orientation matches det(A)

- For odd n:
any choice of viewing vector produces identical orientation sign
- For even n:
even 1-based index of viewing vector produces correct orientation sign

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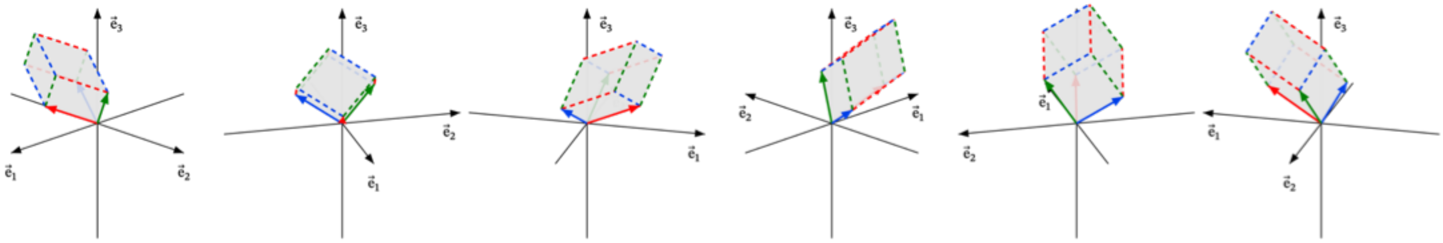
Cofactor expansion visualization

Expanding along col 1 of invertible 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 2 & -3 \\ 2 & 3 & 2 \end{bmatrix}$ det(A) = 55

- det(A) is the signed volume enclosed by the parallelepiped

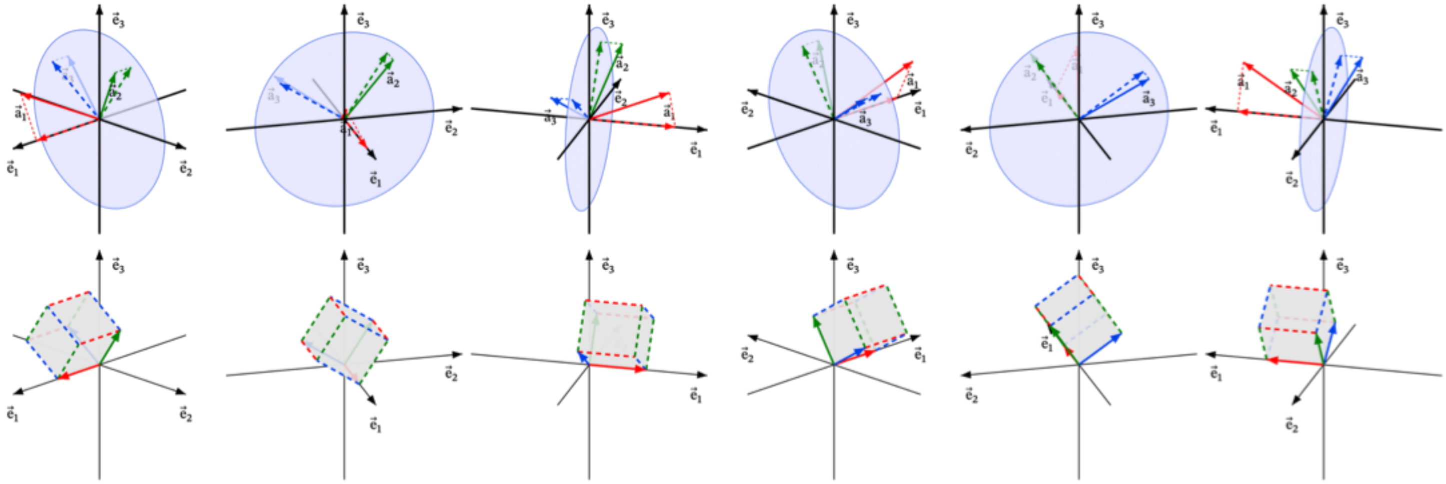


- det(A) can be presented as a sum of 3 terms
- each term is built from one projection of $\vec{a}_1$ and one projected minor
- for term i, $\vec{a}_1$ is projected onto $\vec{e}_i$
- the remaining vectors are projected onto the coordinate plane $\vec{e}_i\perp$
- top-row images show these projections inside $\vec{e}_i\perp$
- bottom-row images show the corresponding 3D parallelepiped volume
- source of cofactor signs for each term is discussed on the next page

$$\text{Term 1: } a_{11} \det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) = 4 \det\left(\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\right) = 52$$

- a_{11} = projection of $\vec{a}_1$ onto $\vec{e}_1$
- $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ = projection of columns $\vec{a}_2$, $\vec{a}_3$ onto span $\{\vec{e}_2, \vec{e}_3\}$

- absolute value equals volume of parallelepiped enclosed by the 3 vectors
- term keeps the same sign

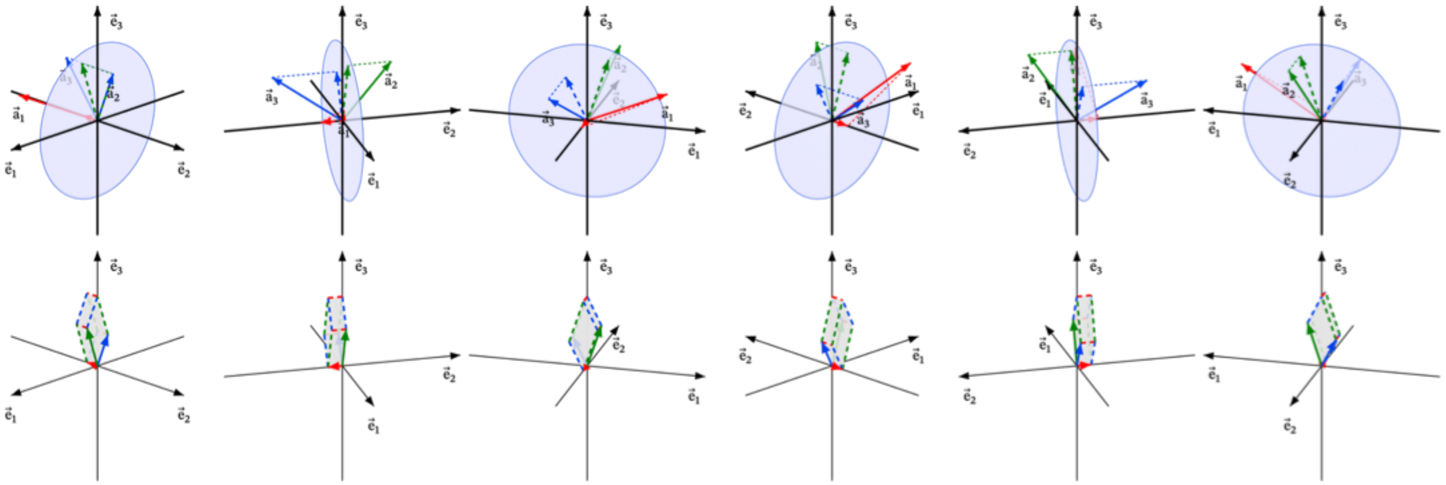


$$\text{Term 2: } a_{21} \det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}\right) = -1 \det\left(\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}\right) = -5$$

- a_{21} = projection of $\vec{a}_1$ onto $\vec{e}_2$

- $\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$ = projection of columns $\vec{a}_2, \vec{a}_3$ onto span $\{ \vec{e}_1, \vec{e}_3 \}$

- absolute value equals volume of parallelepiped enclosed by the 3 vectors
- term sign is negated

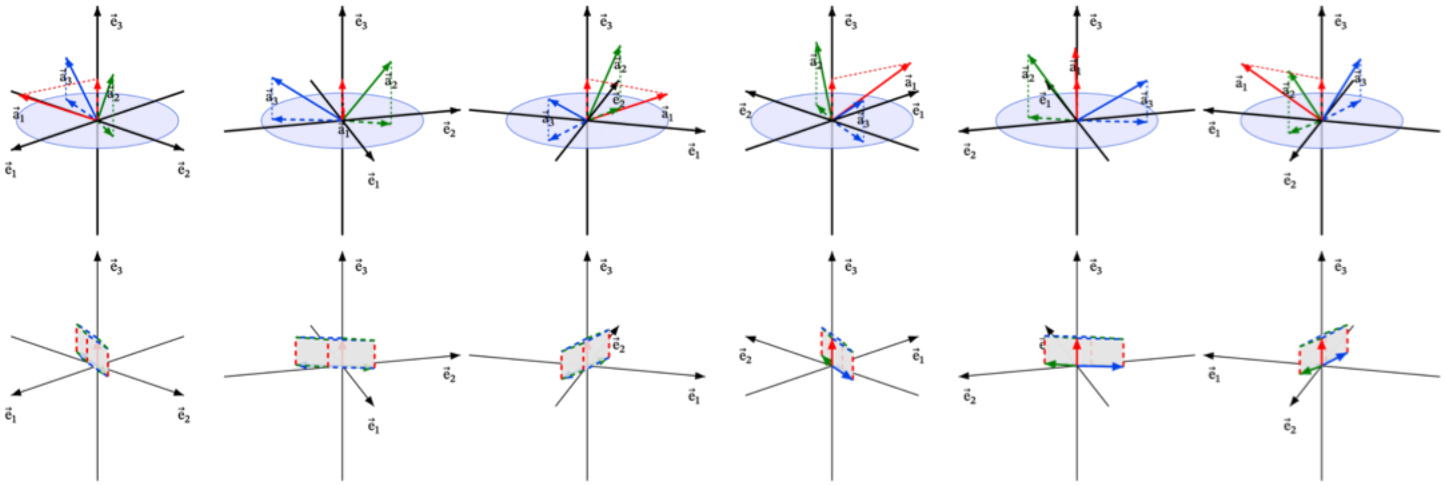


$$\text{Term 3: } a_{31} \det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}\right) = 2 \det\left(\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}\right) = -2$$

- a_{31} = projection of $\vec{a}_1$ onto $\vec{e}_3$

- $\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$ = projection of columns $\vec{a}_2, \vec{a}_3$ onto span $\{ \vec{e}_1, \vec{e}_2 \}$

- absolute value equals volume of parallelepiped enclosed by the 3 vectors
- term keeps the same sign



Term sign visualization in $\mathbb{R}^3$

Expanding along column 1 of 3×3 matrix

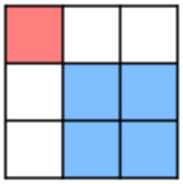
Each step tracks

- the current orientation formed by the pivot row direction, followed by the row directions of the minor submatrix
- the number of swaps needed to obtain the standard order $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$
- even number of swaps → same sign
- odd number of swaps → negated sign

①

- Pivot entry: a_{11}

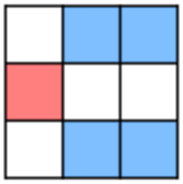
- Current orientation: $\left(\vec{e}_1, \vec{e}_2, \vec{e}_3\right)$
- Swaps needed to obtain standard order: none
- 0 swaps → same sign
- cofactor sign: +



- Pivot entry: a_{21}
- Current orientation: $\left(\vec{e}_2, \vec{e}_1, \vec{e}_3\right)$
- Swaps needed to obtain standard order:
  - $\left(\vec{e}_2, \vec{e}_1, \vec{e}_3\right)$

↓

  - $\left(\vec{e}_1, \vec{e}_2, \vec{e}_3\right)$
- 1 swap → negated sign
- cofactor sign: −



- Pivot entry: a_{31}
- Current orientation: $\left(\vec{e}_3, \vec{e}_1, \vec{e}_2\right)$
- Swaps needed to obtain standard order:
  - $\left(\vec{e}_3, \vec{e}_1, \vec{e}_2\right)$

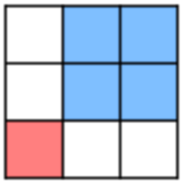
↓

  - $\left(\vec{e}_1, \vec{e}_3, \vec{e}_2\right)$

↓

▸ $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$

- 2 swaps → same sign
- cofactor sign: +



Bottom line:

- for expansion along column 1, term i has current orientation

$$(\vec{e}_i, \vec{e}_1, \dots, \vec{e}_{i-1}, \vec{e}_{i+1}, \dots, \vec{e}_n)$$

- restoring the standard order requires i−1 swaps
- even swaps keep the sign; odd swaps negate the sign
- cofactor sign: $(-1)^{(i-1)} = (-1)^{(i+1)}$

Term sign visualization in $\mathbb{R}^4$

Expanding along column 1 of 4×4 matrix

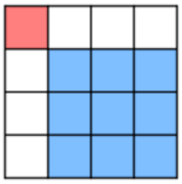
Each step tracks

- the current orientation formed by the pivot row direction, followed by the row directions of the minor submatrix
- the number of swaps needed to obtain the standard order $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$

- even number of swaps → same sign
- odd number of swaps → negated sign

①

- Pivot entry: a_{11}
- Current orientation: $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$
- Swaps needed to obtain standard order: none
- 0 swaps → same sign
- cofactor sign: +



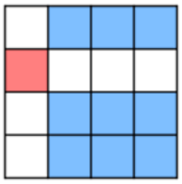
- Pivot entry: a_{21}
- Current orientation: $(\vec{e}_2, \vec{e}_1, \vec{e}_3, \vec{e}_4)$
- Swaps needed to obtain standard order:

‣ $(\vec{e}_2, \vec{e}_1, \vec{e}_3, \vec{e}_4)$

↓

‣ $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$

- 1 swap → negated sign
- cofactor sign: −



- Pivot entry: a_{31}
- Current orientation: $(\vec{e}_3, \vec{e}_1, \vec{e}_2, \vec{e}_4)$

- Swaps needed to obtain standard order:
  - $\left(\vec{e}_3, \vec{e}_1, \vec{e}_2, \vec{e}_4 \right)$

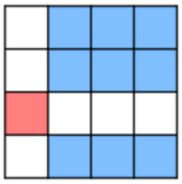
↓

  - $\left(\vec{e}_1, \vec{e}_3, \vec{e}_2, \vec{e}_4 \right)$

↓

  - $\left(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 \right)$

- 2 swaps → same sign
- cofactor sign: +



- Pivot entry: a_{41}
- Current orientation: $\left(\vec{e}_4, \vec{e}_1, \vec{e}_2, \vec{e}_3 \right)$
- Swaps needed to obtain standard order:
  - $\left(\vec{e}_4, \vec{e}_1, \vec{e}_2, \vec{e}_3 \right)$

↓

  - $\left(\vec{e}_1, \vec{e}_4, \vec{e}_2, \vec{e}_3 \right)$

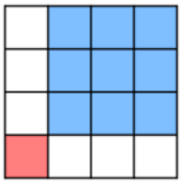
↓

  - $\left(\vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_3 \right)$

↓

  - $\left(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 \right)$

- 3 swaps → negated sign
- cofactor sign: −



Bottom line:

- for expansion along column 1, term i has current orientation

$$\left(\vec{e}_i, \vec{e}_1, \ldots, \vec{e}_{i-1}, \vec{e}_{i+1}, \ldots, \vec{e}_n \right)$$

- restoring the standard order requires i−1 swaps
- even swaps keep the sign; odd swaps negate the sign
- cofactor sign: $(-1)^{(i-1)} = (-1)^{(i+1)}$