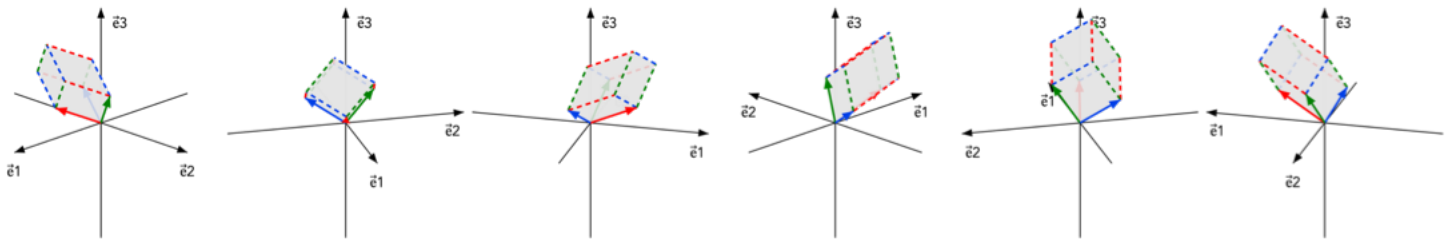


# Sign visualization in 3D ( $\det(A) > 0$ )

Invertible  $3 \times 3$  matrix  $A = [\vec{a}_1 \mid \vec{a}_2 \mid \vec{a}_3] = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 2 & -3 \\ 2 & 3 & 2 \end{bmatrix}$   $\det(A) = 55$

$\det(A)$  is the signed volume enclosed by the parallelepiped:



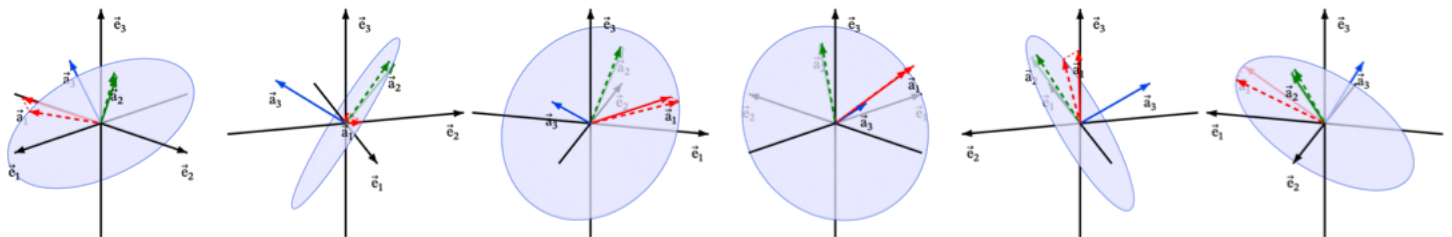
Determination of  $\det(A)$  sign:

①

- project onto  $\vec{a}_3 \perp$
- view from the tip of  $\vec{a}_3$
- counterclockwise order of projections matches the cyclic column order  $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \dots$



$\det(A) > 0$



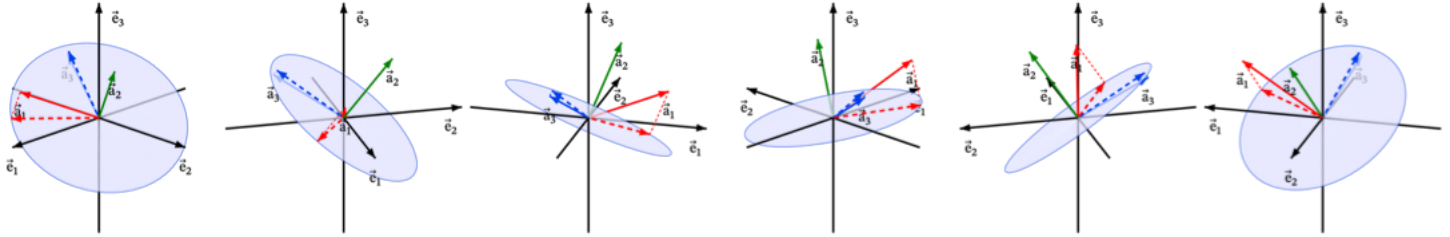
②

- project onto  $\vec{a}_2 \perp$

- view from the tip of  $\vec{a}_2$
- counterclockwise order of projections matches the cyclic column order  $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \dots$



$$\det(A) > 0$$

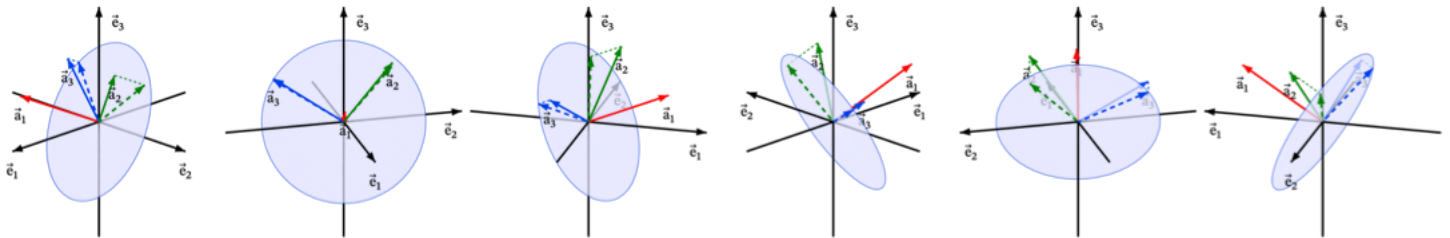


③

- project onto  $\vec{a}_1 \perp$
- view from the tip of  $\vec{a}_1$
- counterclockwise order of projections matches the cyclic column order  $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \dots$



$$\det(A) > 0$$



Ⓐ

How is the projection plane defined?

For each method, choose one column vector as the viewing vector

Then project the two remaining column vectors onto the plane orthogonal to it:

- method ①: project  $\vec{a}_1$  and  $\vec{a}_2$  onto  $\vec{a}_3 \perp$
- method ②: project  $\vec{a}_3$  and  $\vec{a}_1$  onto  $\vec{a}_2 \perp$
- method ③: project  $\vec{a}_2$  and  $\vec{a}_3$  onto  $\vec{a}_1 \perp$

Ⓑ

How is the sign read visually?

Look from the tip of the viewing vector toward the origin

If the projected arrows appear in the cyclic column order

$\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \dots$  then  $\det(A) > 0$

If the order is reversed, then  $\det(A) < 0$

Ⓒ

Why do methods ② and ③ produce results identical to method ①?

Method ②:

Suppose  $B = [ \vec{b}_1 \mid \vec{b}_2 \mid \vec{b}_3 ] = [ \vec{a}_3 \mid \vec{a}_1 \mid \vec{a}_2 ]$

B is obtained from A by 2 column exchanges

↓

$$\det(B) = \det(A)$$

Applying method ① to B means projecting onto  $\vec{b}_3 \perp$

$$\vec{b}_3 = \vec{a}_2$$

↓

method ① for B is identical to method ② for A

Method ③:

Suppose  $C = [ \vec{c}_1 \mid \vec{c}_2 \mid \vec{c}_3 ] = [ \vec{a}_2 \mid \vec{a}_3 \mid \vec{a}_1 ]$

C is obtained from A by 2 column exchanges

↓

$$\det(C) = \det(A)$$

Applying method ① to C means projecting onto  $\vec{c}_3 \perp$

$$\vec{c}_3 = \vec{a}_1$$

↓

method ① for C is identical to method ③ for A

This works directly because  $n = 3$  is odd:  
cyclic column shifts preserve determinant sign in odd dimensions

and reverse determinant sign in even dimensions

Ⓓ

What if  $\det(A) = 0$ ?

after projection onto  $\vec{a}_3^\perp$ , the two projected vectors are linearly dependent



no counterclockwise order is defined

Ⓔ

The visualization reduces the determinant sign problem to reading orientation in the projection plane:

- choose one vector as the viewing vector
- project the other two vectors onto the orthogonal plane
- look from the tip of the viewing vector toward the origin
- compare the visible counterclockwise order with the cyclic column order

If the projected arrows appear in cyclic order,  $\det(A) > 0$

If the order is reversed,  $\det(A) < 0$

If the projected arrows are linearly dependent,  $\det(A) = 0$



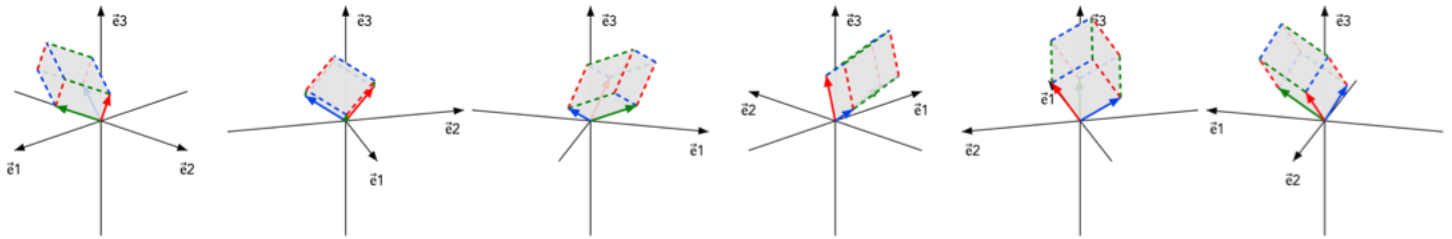
Sign visualization in 3D ( $\det(A) < 0$ )

Invertible  $3 \times 3$  matrix  $A = [\vec{a}_1 \mid \vec{a}_2 \mid \vec{a}_3] =$

1	4	-1
2	-1	-3
3	2	2

$\det(A) = -55$

$\det(A)$  is the signed volume enclosed by the parallelepiped:



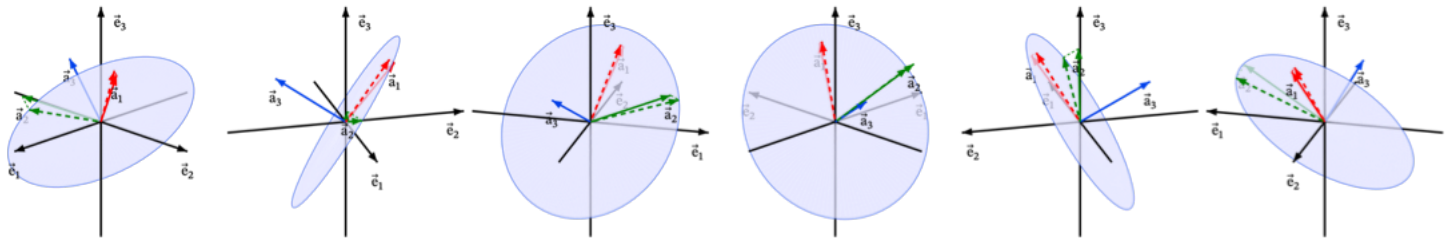
Determination of  $\det(A)$  sign:

①

- project onto  $\vec{a}_3 \perp$
- view from the tip of  $\vec{a}_3$
- counterclockwise order of projections does not match the cyclic column order  $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \dots$



$\det(A) < 0$

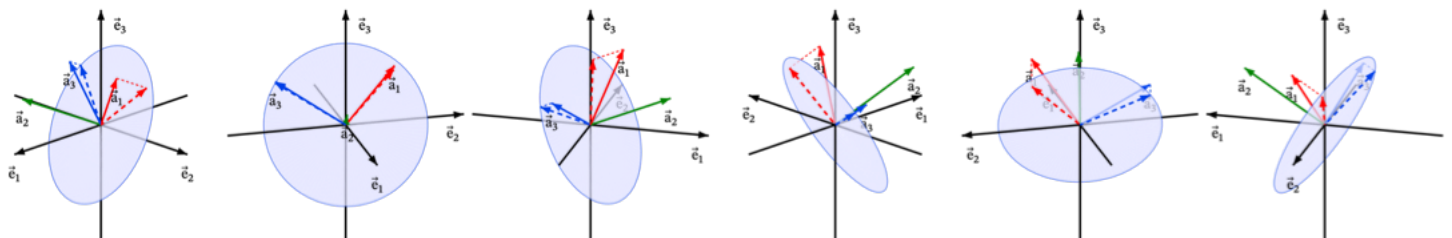


②

- project onto  $\vec{a}_2 \perp$
- view from the tip of  $\vec{a}_2$
- counterclockwise order of projections does not match the cyclic column order  $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \dots$



$\det(A) < 0$

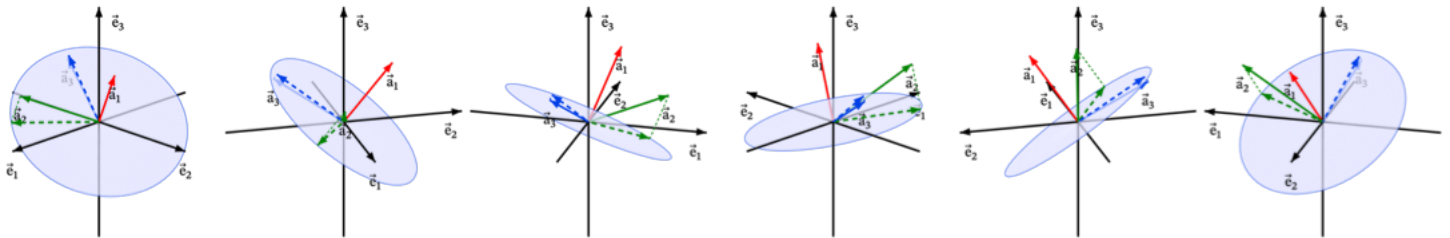


③

- project onto  $\vec{a}_1 \perp$
- view from the tip of  $\vec{a}_1$
- counterclockwise order of projections does not match the cyclic column order  $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \dots$



$\det(A) < 0$



Ⓐ

How is the projection plane defined?

For each method, choose one column vector as the viewing vector

Then project the two remaining column vectors onto the plane orthogonal to it:

- method ①: project  $\vec{a}_1$  and  $\vec{a}_2$  onto  $\vec{a}_3 \perp$
- method ②: project  $\vec{a}_3$  and  $\vec{a}_1$  onto  $\vec{a}_2 \perp$
- method ③: project  $\vec{a}_2$  and  $\vec{a}_3$  onto  $\vec{a}_1 \perp$

Ⓑ

How is the sign read visually?

Look from the tip of the viewing vector toward the origin

If the projected arrows appear in the cyclic column order

$\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_1, \dots$  then  $\det(A) > 0$

If the order is reversed, then  $\det(A) < 0$

Ⓒ

Why do methods ② and ③ produce results identical to method ①?

Method ②:

$$\text{Suppose } B = [\vec{b}_1 \mid \vec{b}_2 \mid \vec{b}_3] = [\vec{a}_3 \mid \vec{a}_1 \mid \vec{a}_2]$$

B is obtained from A by 2 column exchanges



$$\det(B) = \det(A)$$

Applying method ① to B means projecting onto  $\vec{b}_3 \perp$

$$\vec{b}_3 = \vec{a}_2$$



method ① for B is identical to method ② for A

Method ③:

$$\text{Suppose } C = [\vec{c}_1 \mid \vec{c}_2 \mid \vec{c}_3] = [\vec{a}_2 \mid \vec{a}_3 \mid \vec{a}_1]$$

C is obtained from A by 2 column exchanges



$$\det(C) = \det(A)$$

Applying method ① to C means projecting onto  $\vec{c}_3 \perp$

$$\vec{c}_3 = \vec{a}_1$$



method ① for C is identical to method ③ for A

This works directly because  $n = 3$  is odd:  
cyclic column shifts preserve determinant sign in odd dimensions  
and reverse determinant sign in even dimensions

Ⓓ

What if  $\det(A) = 0$ ?

after projection onto  $\vec{a}_3 \perp$ , the two projected vectors are linearly dependent



no counterclockwise order is defined

Ⓔ

The visualization reduces the determinant sign problem

to reading orientation in the projection plane:

- choose one vector as the viewing vector
- project the other two vectors onto the orthogonal plane
- look from the tip of the viewing vector toward the origin
- compare the visible counterclockwise order with the cyclic column order

If the projected arrows appear in cyclic order,  $\det(A) > 0$

If the order is reversed,  $\det(A) < 0$

If the projected arrows are linearly dependent,  $\det(A) = 0$



Determinant sign as a recursive projection rule

4D version:

For  $A = [ \vec{a}_1 \mid \vec{a}_2 \mid \vec{a}_3 \mid \vec{a}_4 ]$ :

①

Start with  $\vec{a}_4$  as the viewing vector:

ⓐ Project  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  onto  $\vec{a}_4^\perp$

this reduces the problem to a 3D orientation

ⓑ choose a projected vector (say  $\vec{a}_3$ ) and project the remaining two onto  $\vec{a}_3^\perp$

we now have a 2D orientation problem where  
the final 2D counterclockwise order matches  $\det(A)$

②

Start with  $\vec{a}_3$  as the viewing vector:

To place  $\vec{a}_3$  last, one column exchange is needed

This introduces a sign flip,  
 so after the same projection steps, the result is the opposite sign of  $\det(A)$

③

Start with  $\vec{a}_2$ :

Placing  $\vec{a}_2$  last requires two exchanges, which preserve the sign

After projecting onto  $\vec{a}_2^\perp$  and following the same steps,  
 the orientation matches  $\det(A)$

- For odd n:  
 any choice of viewing vector produces identical orientation sign
- For even n:  
 even 1-based index of viewing vector produces correct orientation sign

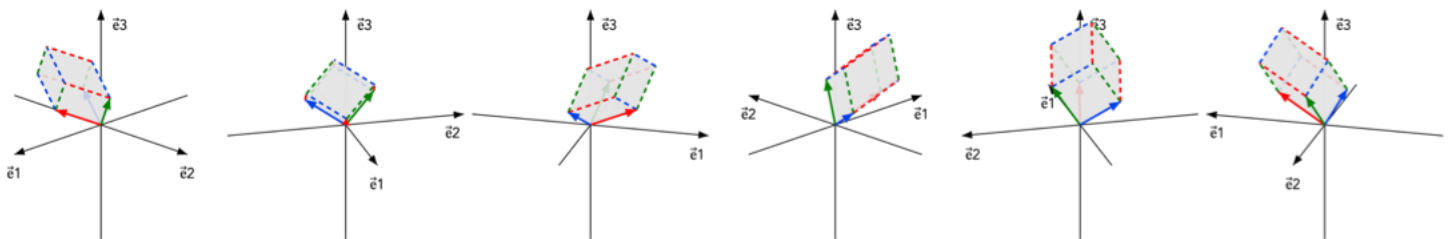


### Cofactor expansion visualization

Expanding along col 1 of invertible  $3 \times 3$  matrix  $A =$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 2 & -3 \\ 2 & 3 & 2 \end{bmatrix} \quad \det(A) = 55$$

- $\det(A)$  is the signed volume enclosed by the parallelepiped



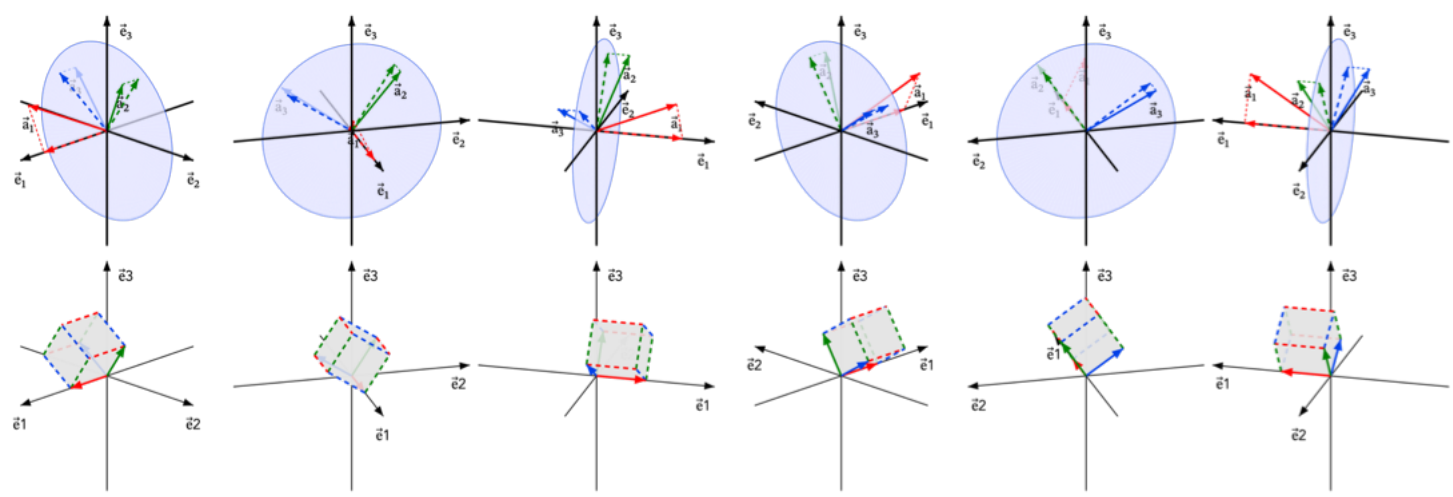
- $\det(A)$  can be presented as a sum of 3 terms
- each term is built from one projection of  $\vec{a}_1$  and one projected minor
  - for term  $i$ ,  $\vec{a}_1$  is projected onto  $\vec{e}_i$
- the remaining vectors are projected onto the coordinate plane  $\vec{e}_i \perp$ 
  - top-row images show these projections inside  $\vec{e}_i \perp$
- bottom-row images show the corresponding 3D parallelepiped volume
  - source of cofactor signs for each term is discussed on the next page

$$\text{Term 1: } a_{11} \det \left( \left[ \begin{array}{c|c} a_{22} & a_{23} \\ \hline a_{32} & a_{33} \end{array} \right] \right) = 4 \det \left( \left[ \begin{array}{c|c} 2 & -3 \\ \hline 3 & 2 \end{array} \right] \right) = 52$$

- $a_{11}$  = projection of  $\vec{a}_1$  onto  $\vec{e}_1$

- $\left[ \begin{array}{c|c} a_{22} & a_{23} \\ \hline a_{32} & a_{33} \end{array} \right]$  = projection of columns  $\vec{a}_2, \vec{a}_3$  onto  $\text{span} \{ \vec{e}_2, \vec{e}_3 \}$

- absolute value equals volume of parallelepiped enclosed by the 3 vectors
  - term keeps the same sign

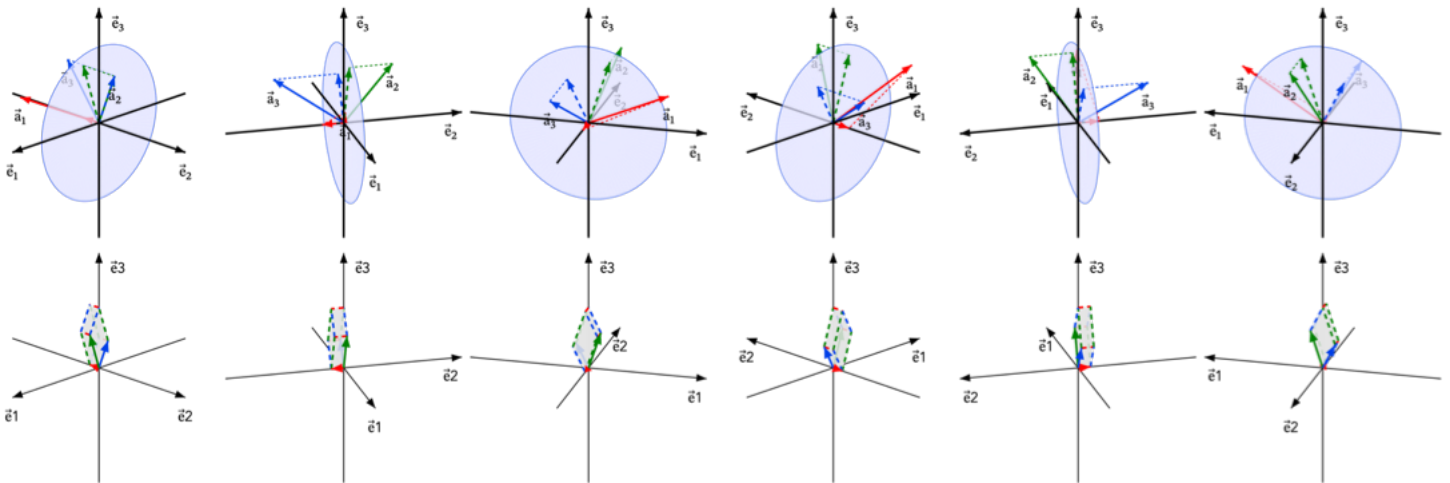


$$\text{Term 2: } a_{21} \det \left( \begin{array}{c|c} a_{12} & a_{13} \\ \hline a_{32} & a_{33} \end{array} \right) = -1 \det \left( \begin{array}{c|c} 1 & -1 \\ \hline 3 & 2 \end{array} \right) = -5$$

- $a_{21}$  = projection of  $\vec{a}_1$  onto  $\vec{e}_2$

- $\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$  = projection of columns  $\vec{a}_2, \vec{a}_3$  onto span  $\{ \vec{e}_1, \vec{e}_3 \}$

- absolute value equals volume of parallelepiped enclosed by the 3 vectors
- term sign is negated

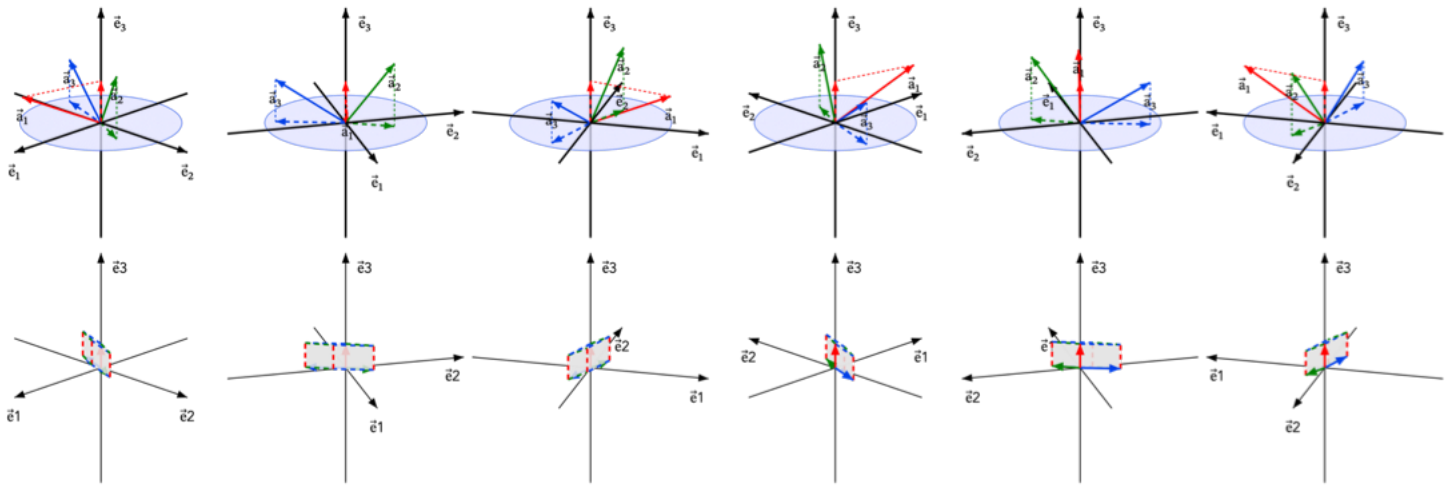


$$\text{Term 3: } a_{31} \det \left( \begin{array}{c|c} a_{12} & a_{13} \\ \hline a_{22} & a_{23} \end{array} \right) = 2 \det \left( \begin{array}{c|c} 1 & -1 \\ \hline 2 & -3 \end{array} \right) = -2$$

- $a_{31}$  = projection of  $\vec{a}_1$  onto  $\vec{e}_3$

- $\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$  = projection of columns  $\vec{a}_2, \vec{a}_3$  onto span  $\{ \vec{e}_1, \vec{e}_2 \}$

- absolute value equals volume of parallelepiped enclosed by the 3 vectors
- term keeps the same sign



Term sign visualization in  $\mathbb{R}^3$

Expanding along column 1 of  $3 \times 3$  matrix

Each step tracks

- the current orientation formed by the pivot row direction, followed by the row directions of the minor submatrix
- the number of swaps needed to obtain the standard order  $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ 
  - even number of swaps  $\rightarrow$  same sign
  - odd number of swaps  $\rightarrow$  negated sign

①

- Pivot entry:  $a_{11}$

- Current orientation:  $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$
- Swaps needed to obtain standard order: none
- 0 swaps  $\rightarrow$  same sign

- cofactor sign: +


- Pivot entry:  $a_{21}$

- Current orientation:  $(\vec{e}_2, \vec{e}_1, \vec{e}_3)$
- Swaps needed to obtain standard order:
  - $(\vec{e}_2, \vec{e}_1, \vec{e}_3)$
  - ↓
  - $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$

- 1 swap  $\rightarrow$  negated sign

- cofactor sign: -


- Pivot entry:  $a_{31}$

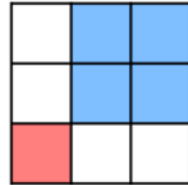
- Current orientation:  $(\vec{e}_3, \vec{e}_1, \vec{e}_2)$
- Swaps needed to obtain standard order:
  - $(\vec{e}_3, \vec{e}_1, \vec{e}_2)$
  - ↓
  - $(\vec{e}_1, \vec{e}_3, \vec{e}_2)$



$$\triangleright (\vec{e}_1, \vec{e}_2, \vec{e}_3)$$

- 2 swaps  $\rightarrow$  same sign

- cofactor sign: +



Bottom line:

- for expansion along column 1, term  $i$  has current orientation

$$(\vec{e}_i, \vec{e}_1, \dots, \vec{e}_{i-1}, \vec{e}_{i+1}, \dots, \vec{e}_n)$$

- restoring the standard order requires  $i-1$  swaps
- even swaps keep the sign; odd swaps negate the sign
- cofactor sign:  $(-1)^{(i-1)} = (-1)^{(i+1)}$



Term sign visualization in  $\mathbb{R}^4$

Expanding along column 1 of  $4 \times 4$  matrix

Each step tracks

- the current orientation formed by the pivot row direction, followed by the row directions of the minor submatrix
- the number of swaps needed to obtain the standard order  $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$

- even number of swaps  $\rightarrow$  same sign
- odd number of swaps  $\rightarrow$  negated sign

①

- Pivot entry:  $a_{11}$
- Current orientation:  $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$
- Swaps needed to obtain standard order: none
- 0 swaps  $\rightarrow$  same sign

- cofactor sign: +


- Pivot entry:  $a_{21}$
- Current orientation:  $(\vec{e}_2, \vec{e}_1, \vec{e}_3, \vec{e}_4)$
- Swaps needed to obtain standard order:

$$\triangleright (\vec{e}_2, \vec{e}_1, \vec{e}_3, \vec{e}_4)$$

$\downarrow$

$$\triangleright (\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$$

- 1 swap  $\rightarrow$  negated sign

- cofactor sign: -


- Pivot entry:  $a_{31}$
- Current orientation:  $(\vec{e}_3, \vec{e}_1, \vec{e}_2, \vec{e}_4)$

- Swaps needed to obtain standard order:

$$\triangleright (\vec{e}_3, \vec{e}_1, \vec{e}_2, \vec{e}_4)$$

↓

$$\triangleright (\vec{e}_1, \vec{e}_3, \vec{e}_2, \vec{e}_4)$$

↓

$$\triangleright (\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$$

- 2 swaps → same sign

- cofactor sign: +


- Pivot entry:  $a_{41}$

- Current orientation:  $(\vec{e}_4, \vec{e}_1, \vec{e}_2, \vec{e}_3)$

- Swaps needed to obtain standard order:

$$\triangleright (\vec{e}_4, \vec{e}_1, \vec{e}_2, \vec{e}_3)$$

↓

$$\triangleright (\vec{e}_1, \vec{e}_4, \vec{e}_2, \vec{e}_3)$$

↓

$$\triangleright (\vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_3)$$

↓

$$\triangleright (\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$$

- 3 swaps → negated sign

- cofactor sign: −


Bottom line:

- for expansion along column 1, term  $i$  has current orientation

$$\left( \vec{e}_i, \vec{e}_1, \dots, \vec{e}_{i-1}, \vec{e}_{i+1}, \dots, \vec{e}_n \right)$$

- restoring the standard order requires  $i-1$  swaps
- even swaps keep the sign; odd swaps negate the sign

- cofactor sign:  $(-1)^{(i-1)} = (-1)^{(i+1)}$

