Illustration of permutation method

Permutation method

(not used in practice for computation, mainly of historical significance)

Consider a 4×4 matrix $M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$

There are 4! = 24 possible ways to reorder column indices { 1, 2, 3, 4 }, called permutations σ

Each permutation has a sign:

positive if there is an even number of inversions in the sequence

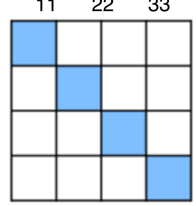
negative if there is an odd number of such inversions

List of permutations of { 1, 2, 3, 4 }:

- σ_1 = { 1, 2, 3, 4 }

inversions: none; sign: positive

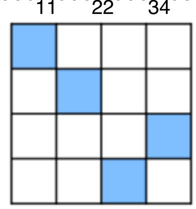
- $m_{11}m_{22}m_{33}m_{44}$



- σ_2 = { 1, 2, 4, 3 }

inversions: 4→3; sign: negative

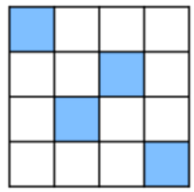
- $m_{11}m_{22}m_{34}m_{43}$



- $\sigma_3 = \{ 1, 3, 2, 4 \}$

inversions: 3→2; sign: negative

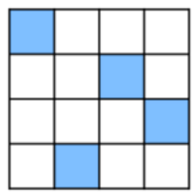
- $m_{11}m_{23}m_{32}m_{44}$



- $\sigma_4 = \{ 1, 3, 4, 2 \}$

inversions: 3→2, 4→2; sign: positive

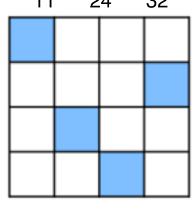
- $m_{11}m_{23}m_{34}m_{42}$



- $\sigma_5 = \{ 1, 4, 2, 3 \}$

inversions: 4→2, 4→3; sign: positive

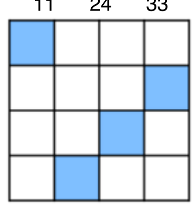
- $m_{11}m_{24}m_{32}m_{43}$



- $\sigma_6 = \{ 1, 4, 3, 2 \}$

inversions: 4→3, 4→2, 3→2; sign: negative

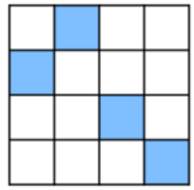
- $m_{11}m_{24}m_{33}m_{42}$



- $\sigma_7 = \{ 2, 1, 3, 4 \}$

inversions: 2→1; sign: negative

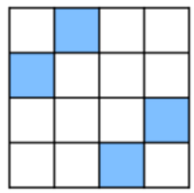
- $m_{12}m_{21}m_{33}m_{44}$



- $\sigma_8 = \{ 2, 1, 4, 3 \}$

inversions: 2→1, 4→3; sign: positive

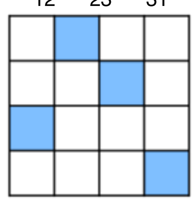
- $m_{12}m_{21}m_{34}m_{43}$



- $\sigma_9 = \{ 2, 3, 1, 4 \}$

inversions: 2→1, 3→1; sign: positive

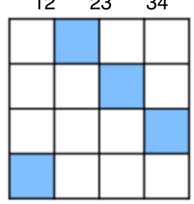
- $m_{12}m_{23}m_{31}m_{44}$



- $\sigma_{10} = \{ 2, 3, 4, 1 \}$

inversions: 2→1, 3→1, 4→1; sign: negative

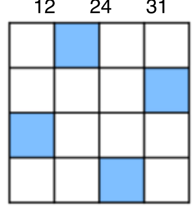
- $m_{12}m_{23}m_{34}m_{41}$



- $\sigma_{11} = \{ 2, 4, 1, 3 \}$

inversions: 2→1, 4→1, 4→3; sign: negative

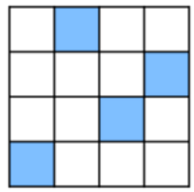
- $m_{12}m_{24}m_{31}m_{43}$



- $\sigma_{12} = \{ 2, 4, 3, 1 \}$

inversions: 2→1, 4→3, 4→1, 3→1; sign: positive

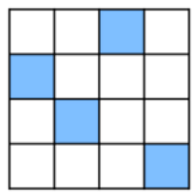
- $m_{12}m_{24}m_{33}m_{41}$



- $\sigma_{13} = \{ 3, 1, 2, 4 \}$

inversions: 3→1, 3→2; sign: positive

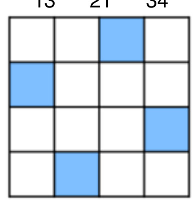
- $m_{13}m_{21}m_{32}m_{44}$



- $\sigma_{14} = \{ 3, 1, 4, 2 \}$

inversions: 3→1, 3→2, 4→2; sign: negative

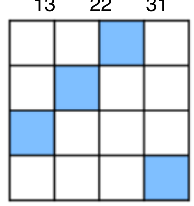
- $m_{13}m_{21}m_{34}m_{42}$



- $\sigma_{15} = \{ 3, 2, 1, 4 \}$

inversions: 3→2, 3→1, 2→1; sign: negative

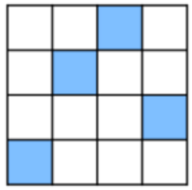
- $m_{13}m_{22}m_{31}m_{44}$



- $\sigma_{16} = \{ 3, 2, 4, 1 \}$

inversions: 3→2, 3→1, 2→1, 4→1; sign: positive

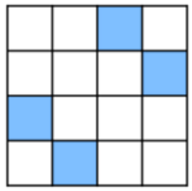
- $m_{13}m_{22}m_{34}m_{41}$



- $\sigma_{17} = \{ 3, 4, 1, 2 \}$

inversions: 3→1, 3→2, 4→1, 4→2; sign: positive

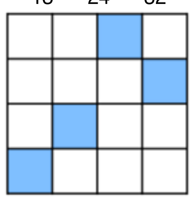
- $m_{13}m_{24}m_{31}m_{42}$



- $\sigma_{18} = \{ 3, 4, 2, 1 \}$

inversions: 3→2, 3→1, 4→2, 4→1, 2→1; sign: negative

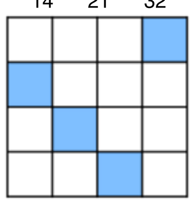
- $m_{13}m_{24}m_{32}m_{41}$



- $\sigma_{19} = \{ 4, 1, 2, 3 \}$

inversions: 4→1, 4→2, 4→3; sign: negative

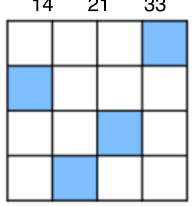
- $m_{14}m_{21}m_{32}m_{43}$



- $\sigma_{20} = \{ 4, 1, 3, 2 \}$

inversions: 4→1, 4→3, 4→2, 3→2; sign: positive

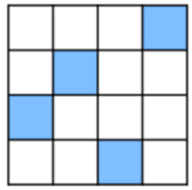
- $m_{14}m_{21}m_{33}m_{42}$



- $\sigma_{21} = \{ 4, 2, 1, 3 \}$

inversions: 4→2, 4→1, 4→3, 2→1; sign: positive

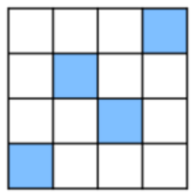
- $m_{14}m_{22}m_{31}m_{43}$



- $\sigma_{22} = \{ 4, 2, 3, 1 \}$

inversions: 4→2, 4→3, 4→1, 2→1, 3→1; sign: negative

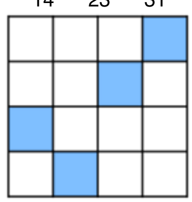
- $m_{14}m_{22}m_{33}m_{41}$



- $\sigma_{23} = \{ 4, 3, 1, 2 \}$

inversions: 4→3, 4→1, 4→2, 3→1, 3→2; sign: negative

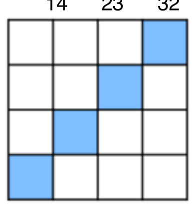
- $m_{14}m_{23}m_{31}m_{42}$



- $\sigma_{24} = \{ 4, 3, 2, 1 \}$

inversions: 4→3, 4→2, 4→1, 3→2, 3→1, 2→1; sign: positive

- $+m_{14}m_{23}m_{32}m_{41}$



- $|M|$ is a sum of $n! = 24$ terms
- Each permutation σ contributes one term & provides the sign of the term
- Each term has n factors

• Each factor is an entry of M chosen as follows:
M entry from column σ(i) in row i

This algorithm generalizes to any n

How this ties with the algebraic definition:

① Swapping rows of M reverses sign of |M|:
swapping rows will reverse the sign of every term

② Scaling one row of M by a constant scales |M| by the same constant:
scaling one row will scale every term by the same scalar

③ Adding a multiple of one row of M to another row leaves |M| unchanged:
adding a multiple of another row produces cancellation
because terms come in pairs with opposite signs
when two rows "compete" for the same column choice

Illustration of cofactor expansion method

Cofactor expansion method
(not used in practice for computation, mainly of historical significance)

① Consider 2×2 matrix $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

$| C | = c_{11}c_{22} - c_{12}c_{21}$

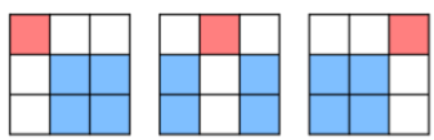
(row reduction is one way to derive this formula)

② Consider 3×3 matrix $M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$

| M | can be computed by expansion along any row or column:
will choose row 1 for this illustration
(for computational efficiency, you can use any row or column with the most 0 entries)

$$m_{11} \times \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} - m_{12} \times \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} + m_{13} \times \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix}$$

same row-1 expansion schematically:



‣ Note alternating signs of each term

‣ Each term uses a minor $|M_{1j}|$; the cofactor is $C_{1j} = (-1)^{(1+j)} |M_{1j}|$

‣ More generally: $| M | = \sum_{j=1}^{n} \left(m_{1j} C_{1j} \right)$

The 3×3 example demonstrates the recursion:
each determinant is reduced to several smaller ones
For larger matrices this quickly becomes impractical,

so determinant computation is done using row reduction

Optional content

How this ties with the algebraic definition:

① Swapping rows of M reverses sign of |M|:
In cofactor expansion, the sign is tied to the row index
Swapping rows moves entries into positions with opposite signs, so every term changes sign

② Scaling one row of M by a constant scales |M| by the same constant:
We can expand along any row or column, thus,
scaling any row will scale every term by the same factor

③ Adding a multiple of one row of M to another row leaves |M| unchanged:

- Calculate | M | by expanding along row 1 (r1):

$$|M| = \sum_{j=1}^{n}\left(m_{1j}\, C_{1j}\right)$$

- Create a new matrix N identical to M except r1(N) = r1(M) + s r2(M)
- Calculate | N | by expanding along row 1:

$$|N| = \sum_{j=1}^{n}\left(\left(m_{1j} + s\, m_{2j}\right) C_{1j}\right) =$$

$$\sum_{j=1}^{n}\left(m_{1j}\, C_{1j}\right) + s \sum_{j=1}^{n}\left(m_{2j} C_{1j}\right)$$

- The first sum is exactly |M|
- The second sum equals | O |, where O is M with row 1 replaced by row 2

In O, rows 1 and 2 are identical

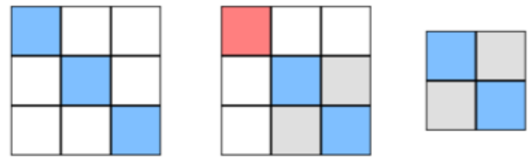
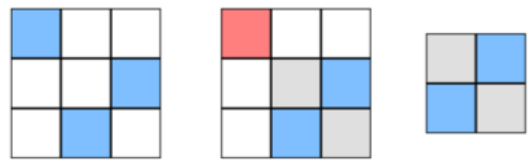
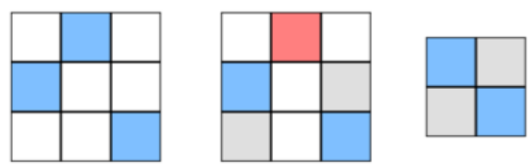
- From (1): swapping rows 1 and 2 negates the determinant, so |swap(O)| = −|O|
But swap(O) = O (matrix is unchanged), so | O | = −| O | ⇒ | O | = 0

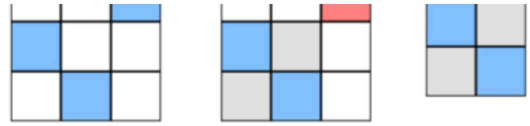
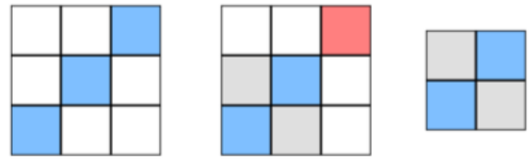
Therefore:
| N | = | M |

One combinatorial algorithm, two names:

Recursive algorithm:
① Iterate over all choices for the first element
② Form the inner set from the remaining elements
③ Repeat ①–② until the inner set has two elements, a and b
④ Return { a, b }, { b, a }

- S_n = all permutation lists of n elements
- S_{n-1} = permutation lists inside the corresponding submatrix
- σ = full list: first choice + remaining choices
- τ = remaining choices inside the submatrix

‣ pivot m_{11} ‣ $\tau = \{2, 3\}$ ‣ submatrix $\begin{bmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{bmatrix}$	‣ $\sigma_1 = \{1, 2, 3\}$ inversions: none; sign: > 0 ‣ $+m_{11}m_{22}m_{33}$ 
‣ pivot m_{11} ‣ $\tau = \{3, 2\}$ ‣ submatrix $\begin{bmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{bmatrix}$	‣ $\sigma_2 = \{1, 3, 2\}$ inversions: 3→2; sign: < 0 ‣ $-m_{11}m_{23}m_{32}$ 
‣ pivot m_{12} ‣ $\tau = \{1, 3\}$ ‣ submatrix $\begin{bmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{bmatrix}$	‣ $\sigma_3 = \{2, 1, 3\}$ inversions: 2→1; sign: < 0 ‣ $-m_{12}m_{21}m_{33}$ 
‣ pivot m_{12} ‣ $\tau = \{3, 1\}$ ‣ submatrix $\begin{bmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{bmatrix}$	‣ $\sigma_4 = \{2, 3, 1\}$ inversions: 2→1, 3→1; sign: > 0 ‣ $+m_{12}m_{23}m_{31}$ 
‣ pivot m_{13} ‣ $\tau = \{1, 2\}$ $\begin{bmatrix} m_{21} & m_{22} \end{bmatrix}$	‣ $\sigma_5 = \{3, 1, 2\}$ inversions: 3→1, 3→2; sign: > 0 ‣ $+m_{13}m_{21}m_{32}$ 

‣ submatrix $\begin{bmatrix} m_{31} & m_{32} \end{bmatrix}$	
‣ pivot m_{13} ‣ $\tau = \{2, 1\}$ ‣ submatrix $\begin{bmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix}$	‣ $\sigma_6 = \{3, 2, 1\}$ inversions: 3→2, 3→1, 2→1; sign: < 0 ‣ $-m_{13}m_{22}m_{31}$ 

One combinatorial algorithm, two names:

Recursive algorithm:
① Iterate over all choices for the first element
② Form the inner set from the remaining elements
③ Repeat ①–② until the inner set has two elements, a and b
④ Return { a, b }, { b, a }

- S_n = all permutation lists of n elements
- S_{n-1} = permutation lists inside the corresponding submatrix
- σ = full list: first choice + remaining choices
- τ = remaining choices inside the submatrix

- pivot m_{11}
- submatrix pivot m_{22}
- $\tau = \{2, 3, 4\}$
- submatrix $\begin{bmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{bmatrix}$

- $\sigma_1 = \{1, 2, 3, 4\}$

inversions: none; sign: > 0

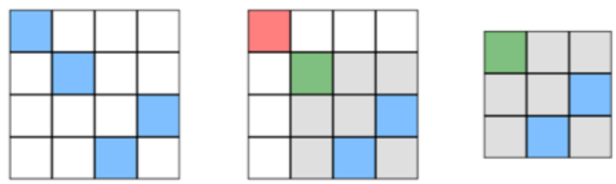
- $+m_{11}m_{22}m_{33}m_{44}$

- pivot m_{11}
- submatrix pivot m_{22}
- $\tau = \{2, 4, 3\}$
- submatrix $\begin{bmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{bmatrix}$

- $\sigma_2 = \{1, 2, 4, 3\}$

inversions: 4→3; sign: < 0

- $-m_{11}m_{22}m_{34}m_{43}$

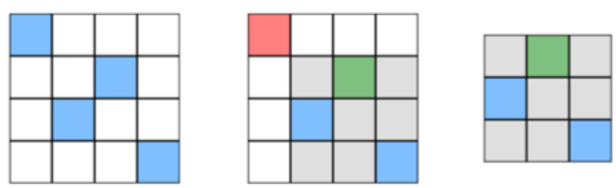


- pivot m_{11}
- submatrix pivot m_{23}
- $\tau = \{3, 2, 4\}$
- submatrix $\begin{bmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{bmatrix}$

- $\sigma_3 = \{1, 3, 2, 4\}$

inversions: 3→2; sign: < 0

- $-m_{11}m_{23}m_{32}m_{44}$

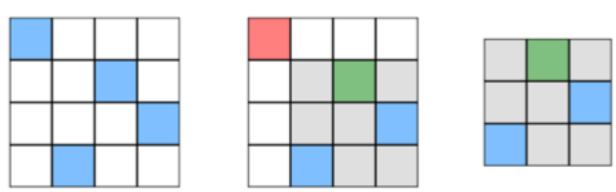


- pivot m_{11}
- submatrix pivot m_{23}
- $\tau = \{3, 4, 2\}$
- submatrix $\begin{bmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{bmatrix}$

- $\sigma_4 = \{1, 3, 4, 2\}$

inversions: 3→2, 4→2; sign: > 0

- $+m_{11}m_{23}m_{34}m_{42}$

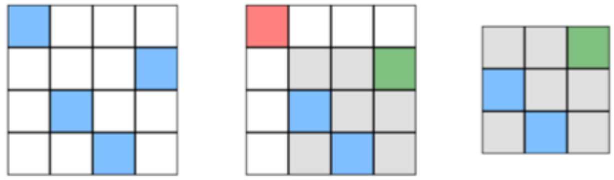


- pivot m_{11}
- submatrix pivot m_{24}
- $\tau = \{4, 2, 3\}$
- submatrix $\begin{bmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{bmatrix}$

- $\sigma_5 = \{1, 4, 2, 3\}$

inversions: 4→2, 4→3; sign: > 0

- $+m_{11}m_{24}m_{32}m_{43}$

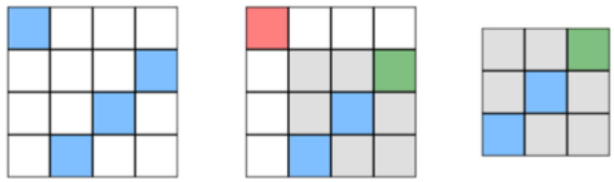


- pivot m_{11}
- submatrix pivot m_{24}
- $\tau = \{4, 3, 2\}$
- submatrix $\begin{bmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{bmatrix}$

- $\sigma_6 = \{1, 4, 3, 2\}$

inversions: 4→3, 4→2, 3→2; sign: < 0

- $-m_{11}m_{24}m_{33}m_{42}$

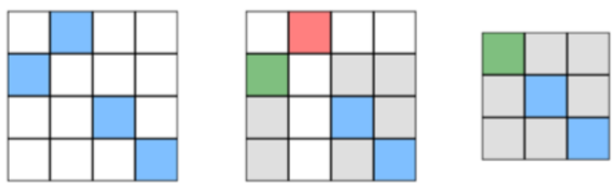


- pivot m_{12}
- submatrix pivot m_{21}
- $\tau = \{1, 3, 4\}$
- submatrix $\begin{bmatrix} m_{21} & m_{23} & m_{24} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{bmatrix}$

- $\sigma_7 = \{2, 1, 3, 4\}$

inversions: 2→1; sign: < 0

- $-m_{12}m_{21}m_{33}m_{44}$

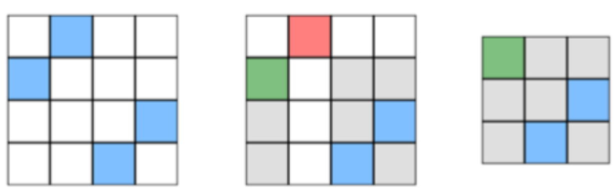


- pivot m_{12}
- submatrix pivot m_{21}
- $\tau = \{1, 4, 3\}$
- submatrix $\begin{bmatrix} m_{21} & m_{23} & m_{24} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{bmatrix}$

- $\sigma_8 = \{2, 1, 4, 3\}$

inversions: 2→1, 4→3; sign: > 0

- $+m_{12}m_{21}m_{34}m_{43}$



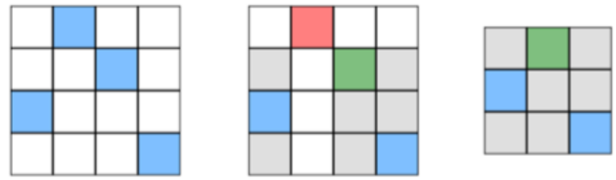
- pivot m_{12}

- submatrix pivot m_{23}
- $\tau = \{3, 1, 4\}$
- submatrix $\begin{bmatrix} m_{21} & m_{23} & m_{24} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{bmatrix}$

- $\sigma_9 = \{2, 3, 1, 4\}$

inversions: 2→1, 3→1; sign: > 0

- $+m_{12}m_{23}m_{31}m_{44}$

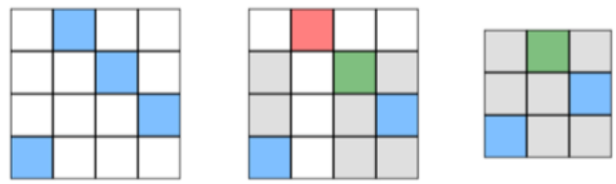


- pivot m_{12}
- submatrix pivot m_{23}
- $\tau = \{3, 4, 1\}$
- submatrix $\begin{bmatrix} m_{21} & m_{23} & m_{24} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{bmatrix}$

- $\sigma_{10} = \{2, 3, 4, 1\}$

inversions: 2→1, 3→1, 4→1; sign: < 0

- $-m_{12}m_{23}m_{34}m_{41}$

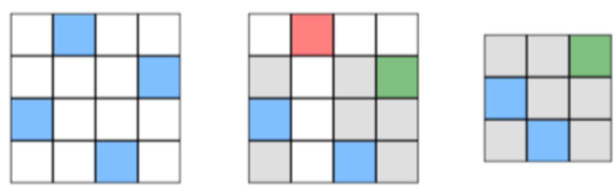


- pivot m_{12}
- submatrix pivot m_{24}
- $\tau = \{4, 1, 3\}$
- submatrix $\begin{bmatrix} m_{21} & m_{23} & m_{24} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{bmatrix}$

- $\sigma_{11} = \{2, 4, 1, 3\}$

inversions: 2→1, 4→1, 4→3; sign: < 0

- $-m_{12}m_{24}m_{31}m_{43}$

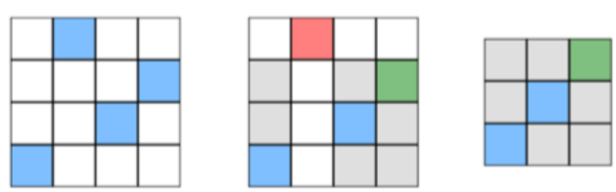


- pivot m_{12}
- submatrix pivot m_{24}
- $\tau = \{4, 3, 1\}$
- submatrix $\begin{bmatrix} m_{21} & m_{23} & m_{24} \\ m_{31} & m_{33} & m_{34} \\ m_{41} & m_{43} & m_{44} \end{bmatrix}$

- $\sigma_{12} = \{2, 4, 3, 1\}$

inversions: 2→1, 4→3, 4→1, 3→1; sign: > 0

- $+m_{12}m_{24}m_{33}m_{41}$



- pivot m_{13}

- $\sigma_{13} = \{3, 1, 2, 4\}$

- submatrix pivot m_{21}
- $\tau = \{1, 2, 4\}$
- submatrix $\begin{bmatrix} m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \\ m_{41} & m_{42} & m_{44} \end{bmatrix}$

inversions: 3→1, 3→2; sign: > 0

- $+m_{13}m_{21}m_{32}m_{44}$

- pivot m_{13}
- submatrix pivot m_{21}
- $\tau = \{1, 4, 2\}$
- submatrix $\begin{bmatrix} m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \\ m_{41} & m_{42} & m_{44} \end{bmatrix}$

- $\sigma_{14} = \{3, 1, 4, 2\}$

inversions: 3→1, 3→2, 4→2; sign: < 0

- $-m_{13}m_{21}m_{34}m_{42}$

- pivot m_{13}
- submatrix pivot m_{22}
- $\tau = \{2, 1, 4\}$
- submatrix $\begin{bmatrix} m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \\ m_{41} & m_{42} & m_{44} \end{bmatrix}$

- $\sigma_{15} = \{3, 2, 1, 4\}$

inversions: 3→2, 3→1, 2→1; sign: < 0

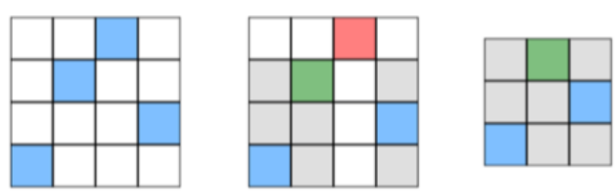
- $-m_{13}m_{22}m_{31}m_{44}$

- pivot m_{13}
- submatrix pivot m_{22}
- $\tau = \{2, 4, 1\}$
- submatrix $\begin{bmatrix} m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \\ m_{41} & m_{42} & m_{44} \end{bmatrix}$

- $\sigma_{16} = \{3, 2, 4, 1\}$

inversions: 3→2, 3→1, 2→1, 4→1; sign: > 0

- $+m_{13}m_{22}m_{34}m_{41}$



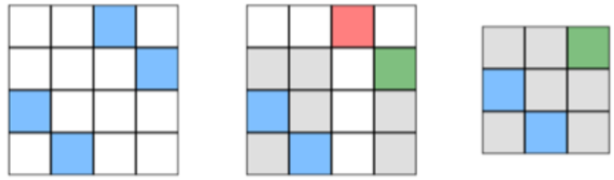
- pivot m_{13}
- submatrix pivot m_{24}

- $\sigma_{17} = \{3, 4, 1, 2\}$

- $\tau = \{4, 1, 2\}$
- submatrix $\begin{bmatrix} m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \\ m_{41} & m_{42} & m_{44} \end{bmatrix}$

inversions: 3→1, 3→2, 4→1, 4→2; sign: > 0

- $+m_{13}m_{24}m_{31}m_{42}$



- pivot m_{13}
- submatrix pivot m_{24}
- $\tau = \{4, 2, 1\}$
- submatrix $\begin{bmatrix} m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \\ m_{41} & m_{42} & m_{44} \end{bmatrix}$

- $\sigma_{18} = \{3, 4, 2, 1\}$

inversions: 3→2, 3→1, 4→2, 4→1, 2→1; sign: < 0

- $-m_{13}m_{24}m_{32}m_{41}$

- pivot m_{14}
- submatrix pivot m_{21}
- $\tau = \{1, 2, 3\}$
- submatrix $\begin{bmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix}$

- $\sigma_{19} = \{4, 1, 2, 3\}$

inversions: 4→1, 4→2, 4→3; sign: < 0

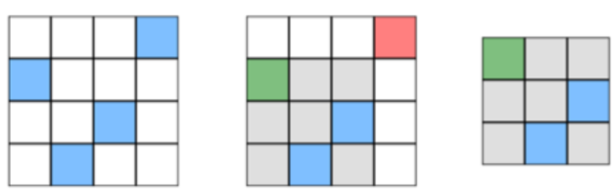
- $-m_{14}m_{21}m_{32}m_{43}$

- pivot m_{14}
- submatrix pivot m_{21}
- $\tau = \{1, 3, 2\}$
- submatrix $\begin{bmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix}$

- $\sigma_{20} = \{4, 1, 3, 2\}$

inversions: 4→1, 4→3, 4→2, 3→2; sign: > 0

- $+m_{14}m_{21}m_{33}m_{42}$



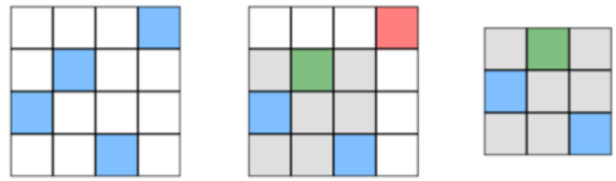
- pivot m_{14}
- submatrix pivot m_{22}
- $\tau = \{2, 1, 3\}$

- $\sigma_{21} = \{4, 2, 1, 3\}$

inversions: 4→2, 4→1, 4→3, 2→1; sign: > 0

‣ submatrix $\begin{bmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix}$

‣ $+m_{14}m_{22}m_{31}m_{43}$



‣ pivot m_{14}

‣ submatrix pivot m_{22}

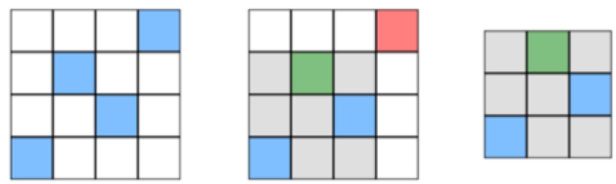
‣ $\tau = \{ 2, 3, 1 \}$

‣ submatrix $\begin{bmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix}$

‣ $\sigma_{22} = \{ 4, 2, 3, 1 \}$

inversions: 4→2, 4→3, 4→1, 2→1, 3→1; sign: < 0

‣ $-m_{14}m_{22}m_{33}m_{41}$



‣ pivot m_{14}

‣ submatrix pivot m_{23}

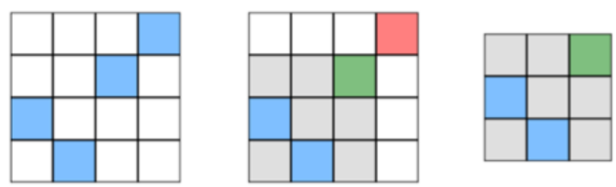
‣ $\tau = \{ 3, 1, 2 \}$

‣ submatrix $\begin{bmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix}$

‣ $\sigma_{23} = \{ 4, 3, 1, 2 \}$

inversions: 4→3, 4→1, 4→2, 3→1, 3→2; sign: < 0

‣ $-m_{14}m_{23}m_{31}m_{42}$



‣ pivot m_{14}

‣ submatrix pivot m_{23}

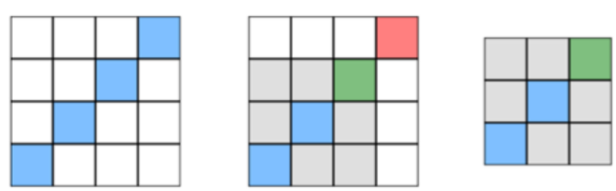
‣ $\tau = \{ 3, 2, 1 \}$

‣ submatrix $\begin{bmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix}$

‣ $\sigma_{24} = \{ 4, 3, 2, 1 \}$

inversions: 4→3, 4→2, 4→1, 3→2, 3→1, 2→1; sign: > 0

‣ $+m_{14}m_{23}m_{32}m_{41}$

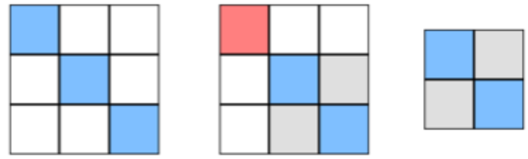


- first-row factor b_{11}
- $\sigma = (2, 3)$
- minor matrix $\begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix}$

- $\tau_1 = (1, 2, 3)$

inversions: none; $\text{sgn}(\tau_1) = +1$

- $+b_{11}b_{22}b_{33}$

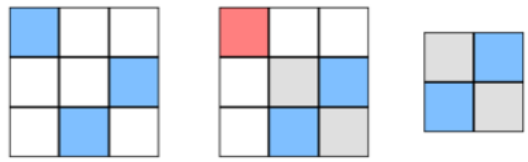


- first-row factor b_{11}
- $\sigma = (3, 2)$
- minor matrix $\begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix}$

- $\tau_2 = (1, 3, 2)$

inversions: $3 \rightarrow 2$; $\text{sgn}(\tau_2) = -1$

- $-b_{11}b_{23}b_{32}$

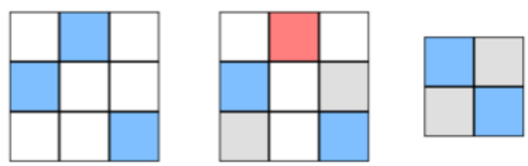


- first-row factor b_{12}
- $\sigma = (1, 3)$
- minor matrix $\begin{bmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{bmatrix}$

- $\tau_3 = (2, 1, 3)$

inversions: $2 \rightarrow 1$; $\text{sgn}(\tau_3) = -1$

- $-b_{12}b_{21}b_{33}$



- first-row factor b_{12}
- $\sigma = (3, 1)$
- minor matrix $\begin{bmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{bmatrix}$

- $\tau_4 = (2, 3, 1)$

inversions: $2 \rightarrow 1$, $3 \rightarrow 1$; $\text{sgn}(\tau_4) = +1$

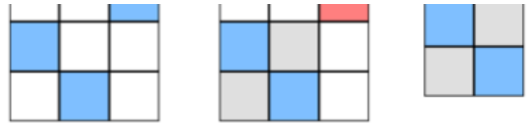
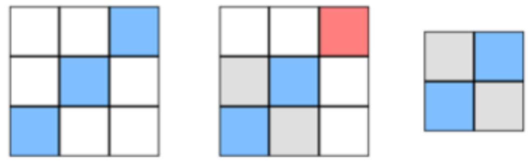
- $+b_{12}b_{23}b_{31}$

- first-row factor b_{13}
- $\sigma = (1, 2)$
- $\begin{bmatrix} b_{21} & b_{22} \end{bmatrix}$

- $\tau_5 = (3, 1, 2)$

inversions: $3 \rightarrow 1$, $3 \rightarrow 2$; $\text{sgn}(\tau_5) = +1$

- $+b_{13}b_{21}b_{32}$

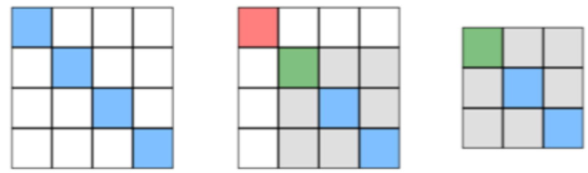
‣ minor matrix $\begin{bmatrix} b_{31} & b_{32} \end{bmatrix}$	
‣ first-row factor b_{13} ‣ $\sigma = (2, 1)$ ‣ minor matrix $\begin{bmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$	‣ $\tau_6 = (3, 2, 1)$ inversions: $3 \rightarrow 2$, $3 \rightarrow 1$, $2 \rightarrow 1$; $\mathrm{sgn}(\tau_6) = -1$ ‣ $-b_{13}b_{22}b_{31}$ 

- first-row factor b_{11}
- minor matrix pivot b_{22}
- $\sigma = (2, 3, 4)$
- minor matrix $\begin{bmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{bmatrix}$

- $\tau_1 = (1, 2, 3, 4)$

inversions: none; $\mathrm{sgn}(\tau_1) = +1$

- $+b_{11}b_{22}b_{33}b_{44}$

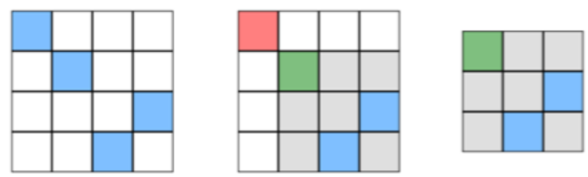


- first-row factor b_{11}
- minor matrix pivot b_{22}
- $\sigma = (2, 4, 3)$
- minor matrix $\begin{bmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{bmatrix}$

- $\tau_2 = (1, 2, 4, 3)$

inversions: $4 \to 3$; $\mathrm{sgn}(\tau_2) = -1$

- $-b_{11}b_{22}b_{34}b_{43}$

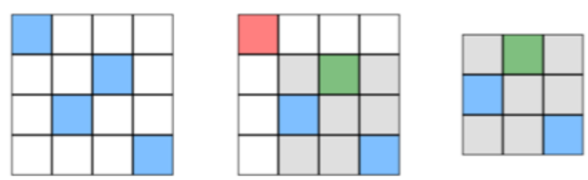


- first-row factor b_{11}
- minor matrix pivot b_{23}
- $\sigma = (3, 2, 4)$
- minor matrix $\begin{bmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{bmatrix}$

- $\tau_3 = (1, 3, 2, 4)$

inversions: $3 \to 2$; $\mathrm{sgn}(\tau_3) = -1$

- $-b_{11}b_{23}b_{32}b_{44}$

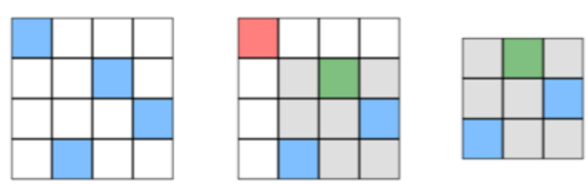


- first-row factor b_{11}
- minor matrix pivot b_{23}
- $\sigma = (3, 4, 2)$
- minor matrix $\begin{bmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{bmatrix}$

- $\tau_4 = (1, 3, 4, 2)$

inversions: $3 \to 2$, $4 \to 2$; $\mathrm{sgn}(\tau_4) = +1$

- $+b_{11}b_{23}b_{34}b_{42}$



- first-row factor b_{11}
- minor matrix pivot b_{24}
- $\sigma = (4, 2, 3)$

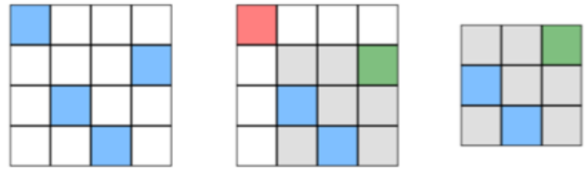
- $\tau_5 = (1, 4, 2, 3)$

inversions: $4 \to 2$, $4 \to 3$; $\mathrm{sgn}(\tau_5) = +1$

‣ σ = (4, 2, 3)

‣ minor matrix $\begin{bmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{bmatrix}$

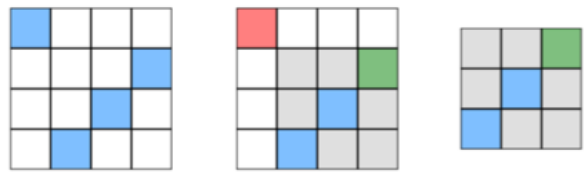
‣ $+b_{11}b_{24}b_{32}b_{43}$



‣ first-row factor b_{11}
‣ minor matrix pivot b_{24}
‣ σ = (4, 3, 2)

‣ minor matrix $\begin{bmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{bmatrix}$

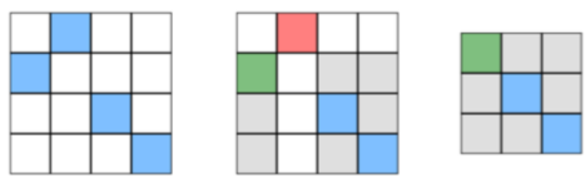
‣ $\tau_6 = (1, 4, 3, 2)$
inversions: 4→3, 4→2, 3→2; $\mathrm{sgn}(\tau_6) = -1$
‣ $-b_{11}b_{24}b_{33}b_{42}$



‣ first-row factor b_{12}
‣ minor matrix pivot b_{21}
‣ σ = (1, 3, 4)

‣ minor matrix $\begin{bmatrix} b_{21} & b_{23} & b_{24} \\ b_{31} & b_{33} & b_{34} \\ b_{41} & b_{43} & b_{44} \end{bmatrix}$

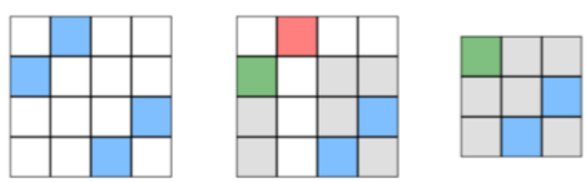
‣ $\tau_7 = (2, 1, 3, 4)$
inversions: 2→1; $\mathrm{sgn}(\tau_7) = -1$
‣ $-b_{12}b_{21}b_{33}b_{44}$



‣ first-row factor b_{12}
‣ minor matrix pivot b_{21}
‣ σ = (1, 4, 3)

‣ minor matrix $\begin{bmatrix} b_{21} & b_{23} & b_{24} \\ b_{31} & b_{33} & b_{34} \\ b_{41} & b_{43} & b_{44} \end{bmatrix}$

‣ $\tau_8 = (2, 1, 4, 3)$
inversions: 2→1, 4→3; $\mathrm{sgn}(\tau_8) = +1$
‣ $+b_{12}b_{21}b_{34}b_{43}$

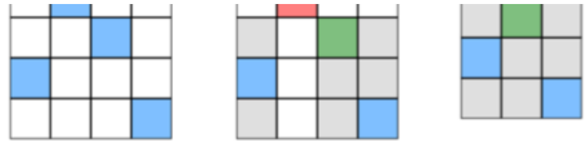


‣ first-row factor b_{12}
‣ minor matrix pivot b_{23}
‣ σ = (3, 1, 4)

‣ $\tau_9 = (2, 3, 1, 4)$
inversions: 2→1, 3→1; $\mathrm{sgn}(\tau_9) = +1$
‣ $+b_{12}b_{23}b_{31}b_{44}$

$\begin{bmatrix} b_{21} & b_{23} & b_{24} \end{bmatrix}$

- minor matrix $\begin{bmatrix} b_{21} & b_{23} & b_{24} \\ b_{31} & b_{33} & b_{34} \\ b_{41} & b_{43} & b_{44} \end{bmatrix}$

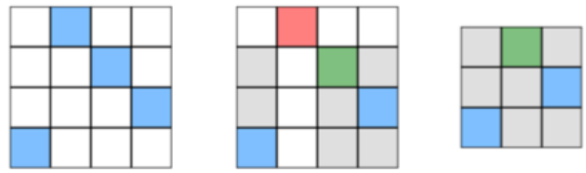


- first-row factor b_{12}
- minor matrix pivot b_{23}
- $\sigma = (3, 4, 1)$
- minor matrix $\begin{bmatrix} b_{21} & b_{23} & b_{24} \\ b_{31} & b_{33} & b_{34} \\ b_{41} & b_{43} & b_{44} \end{bmatrix}$

- $\tau_{10} = (2, 3, 4, 1)$

inversions: 2→1, 3→1, 4→1; $\operatorname{sgn}(\tau_{10}) = -1$

- $-b_{12}b_{23}b_{34}b_{41}$

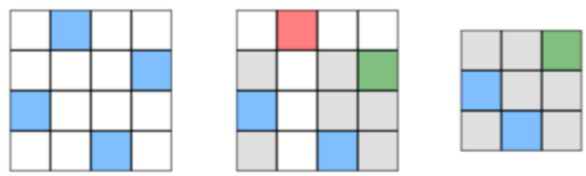


- first-row factor b_{12}
- minor matrix pivot b_{24}
- $\sigma = (4, 1, 3)$
- minor matrix $\begin{bmatrix} b_{21} & b_{23} & b_{24} \\ b_{31} & b_{33} & b_{34} \\ b_{41} & b_{43} & b_{44} \end{bmatrix}$

- $\tau_{11} = (2, 4, 1, 3)$

inversions: 2→1, 4→1, 4→3; $\operatorname{sgn}(\tau_{11}) = -1$

- $-b_{12}b_{24}b_{31}b_{43}$

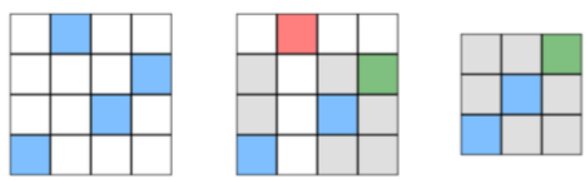


- first-row factor b_{12}
- minor matrix pivot b_{24}
- $\sigma = (4, 3, 1)$
- minor matrix $\begin{bmatrix} b_{21} & b_{23} & b_{24} \\ b_{31} & b_{33} & b_{34} \\ b_{41} & b_{43} & b_{44} \end{bmatrix}$

- $\tau_{12} = (2, 4, 3, 1)$

inversions: 2→1, 4→3, 4→1, 3→1; $\operatorname{sgn}(\tau_{12}) = +1$

- $+b_{12}b_{24}b_{33}b_{41}$

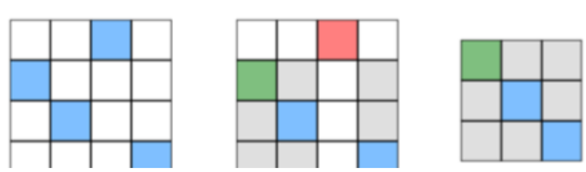


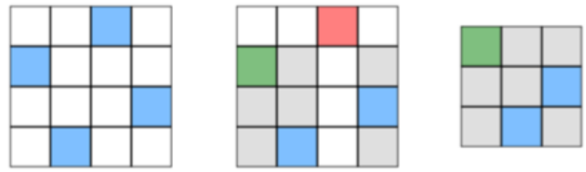
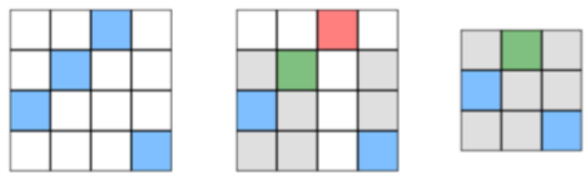
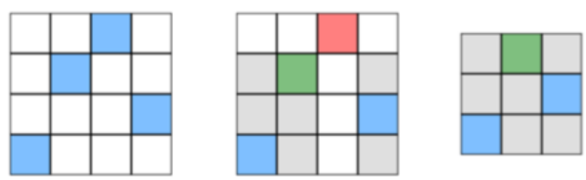
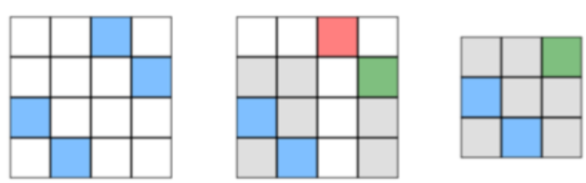
- first-row factor b_{13}
- minor matrix pivot b_{21}
- $\sigma = (1, 2, 4)$
- minor matrix $\begin{bmatrix} b_{21} & b_{22} & b_{24} \\ b_{31} & b_{32} & b_{34} \\ b & b & b \end{bmatrix}$

- $\tau_{13} = (3, 1, 2, 4)$

inversions: 3→1, 3→2; $\operatorname{sgn}(\tau_{13}) = +1$

- $+b_{13}b_{21}b_{32}b_{44}$



$\begin{bmatrix} \cdots & \cdots & \cdots \\ b_{41} & b_{42} & b_{44} \end{bmatrix}$	
‣ first-row factor b_{13} ‣ minor matrix pivot b_{21} ‣ $\sigma = (1, 4, 2)$ ‣ minor matrix $\begin{bmatrix} b_{21} & b_{22} & b_{24} \\ b_{31} & b_{32} & b_{34} \\ b_{41} & b_{42} & b_{44} \end{bmatrix}$	‣ $\tau_{14} = (3, 1, 4, 2)$ inversions: $3 \rightarrow 1$, $3 \rightarrow 2$, $4 \rightarrow 2$; $\operatorname{sgn}(\tau_{14}) = -1$ ‣ $-b_{13}b_{21}b_{34}b_{42}$ 
‣ first-row factor b_{13} ‣ minor matrix pivot b_{22} ‣ $\sigma = (2, 1, 4)$ ‣ minor matrix $\begin{bmatrix} b_{21} & b_{22} & b_{24} \\ b_{31} & b_{32} & b_{34} \\ b_{41} & b_{42} & b_{44} \end{bmatrix}$	‣ $\tau_{15} = (3, 2, 1, 4)$ inversions: $3 \rightarrow 2$, $3 \rightarrow 1$, $2 \rightarrow 1$; $\operatorname{sgn}(\tau_{15}) = -1$ ‣ $-b_{13}b_{22}b_{31}b_{44}$ 
‣ first-row factor b_{13} ‣ minor matrix pivot b_{22} ‣ $\sigma = (2, 4, 1)$ ‣ minor matrix $\begin{bmatrix} b_{21} & b_{22} & b_{24} \\ b_{31} & b_{32} & b_{34} \\ b_{41} & b_{42} & b_{44} \end{bmatrix}$	‣ $\tau_{16} = (3, 2, 4, 1)$ inversions: $3 \rightarrow 2$, $3 \rightarrow 1$, $2 \rightarrow 1$, $4 \rightarrow 1$; $\operatorname{sgn}(\tau_{16}) = +1$ ‣ $+b_{13}b_{22}b_{34}b_{41}$ 
‣ first-row factor b_{13} ‣ minor matrix pivot b_{24} ‣ $\sigma = (4, 1, 2)$ ‣ minor matrix $\begin{bmatrix} b_{21} & b_{22} & b_{24} \\ b_{31} & b_{32} & b_{34} \\ b_{41} & b_{42} & b_{44} \end{bmatrix}$	‣ $\tau_{17} = (3, 4, 1, 2)$ inversions: $3 \rightarrow 1$, $3 \rightarrow 2$, $4 \rightarrow 1$, $4 \rightarrow 2$; $\operatorname{sgn}(\tau_{17}) = +1$ ‣ $+b_{13}b_{24}b_{31}b_{42}$ 
‣ first-row factor b	

- first-row factor b_{13}
- minor matrix pivot b_{24}
- $\sigma = (4, 2, 1)$
- minor matrix $\begin{bmatrix} b_{21} & b_{22} & b_{24} \\ b_{31} & b_{32} & b_{34} \\ b_{41} & b_{42} & b_{44} \end{bmatrix}$

- $\tau_{18} = (3, 4, 2, 1)$

inversions: 3→2, 3→1, 4→2, 4→1, 2→1; $\mathrm{sgn}(\tau_{18}) = -1$

- $-b_{13}b_{24}b_{32}b_{41}$

- first-row factor b_{14}
- minor matrix pivot b_{21}
- $\sigma = (1, 2, 3)$
- minor matrix $\begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$

- $\tau_{19} = (4, 1, 2, 3)$

inversions: 4→1, 4→2, 4→3; $\mathrm{sgn}(\tau_{19}) = -1$

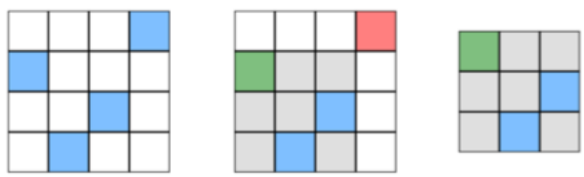
- $-b_{14}b_{21}b_{32}b_{43}$

- first-row factor b_{14}
- minor matrix pivot b_{21}
- $\sigma = (1, 3, 2)$
- minor matrix $\begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$

- $\tau_{20} = (4, 1, 3, 2)$

inversions: 4→1, 4→3, 4→2, 3→2; $\mathrm{sgn}(\tau_{20}) = +1$

- $+b_{14}b_{21}b_{33}b_{42}$

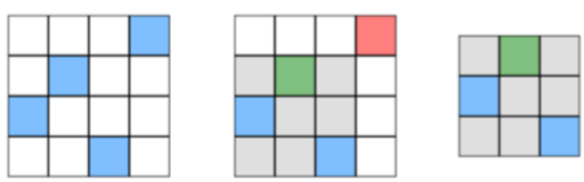


- first-row factor b_{14}
- minor matrix pivot b_{22}
- $\sigma = (2, 1, 3)$
- minor matrix $\begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$

- $\tau_{21} = (4, 2, 1, 3)$

inversions: 4→2, 4→1, 4→3, 2→1; $\mathrm{sgn}(\tau_{21}) = +1$

- $+b_{14}b_{22}b_{31}b_{43}$

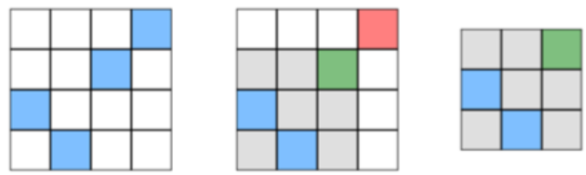


- first-row factor b_{14}
- minor matrix pivot b_{22}
- $\sigma = (2, 3, 1)$

- $\tau_{22} = (4, 2, 3, 1)$

inversions: 4→2, 4→3, 4→1, 2→1, 3→1; $\mathrm{sgn}(\tau_{22}) = -1$

- -b b b b

‣ minor matrix $\begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$	$b_{14}b_{22}b_{33}b_{41}$ 
‣ first-row factor b_{14} ‣ minor matrix pivot b_{23} ‣ $\sigma = (3, 1, 2)$ ‣ minor matrix $\begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$	‣ $\tau_{23} = (4, 3, 1, 2)$ inversions: 4→3, 4→1, 4→2, 3→1, 3→2; $\text{sgn}(\tau_{23}) = -1$ ‣ $-b_{14}b_{23}b_{31}b_{42}$ 
‣ first-row factor b_{14} ‣ minor matrix pivot b_{23} ‣ $\sigma = (3, 2, 1)$ ‣ minor matrix $\begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$	‣ $\tau_{24} = (4, 3, 2, 1)$ inversions: 4→3, 4→2, 4→1, 3→2, 3→1, 2→1; $\text{sgn}(\tau_{24}) = +1$ ‣ $+b_{14}b_{23}b_{32}b_{41}$