Introduction

The cross product is a specifically 3D function that
• inputs two vectors $\vec{k}$ and $\vec{v}$
• outputs one vector $\vec{k} \times \vec{v}$

• The direction $\hat{k}$ determines the plane $\hat{k}^{\perp}$ and the 90° rotation in that plane.
• The length $|\vec{k}|$ scales the result

↓

If $\vec{k} = \vec{0}$, then $\vec{k} \times \vec{v} = \vec{0}$.

So $[\vec{k}]_{\times}$ is defined by the following sequence:

① remove the component of $\vec{v}$ parallel to $\hat{k}$
② rotate the remaining component by 90° in $\hat{k}^{\perp}$
③ return the result to $\mathbb{R}^3$
④ scale by $|\vec{k}|$

For a unit vector $\hat{k}$, step ④ has no effect.
So we first derive the unit matrix $[\hat{k}]_{\times}$.

Ⓐ

For unit $\hat{k}$, $[\hat{k}]_{\times}$ is the composition
① project onto $\hat{k}^{\perp}$
② rotate by 90° in $\hat{k}^{\perp}$
③ return to $\mathbb{R}^3$

Ⓑ

We derive the entries of $[\hat{k}]_{\times}$ by
① changing basis so that $\hat{k}$ becomes $\vec{e}_1$
② solving the simple $\vec{e}_1$ case
③ changing back to the standard basis

Ⓒ

For a general nonzero vector $\vec{k}$,

$$[\,\vec{k}\,]\times = |\,\vec{k}\,|\,[\,\hat{k}\,]\times$$

Ⓓ

The length formula and other consequences will be derived later.

Cross product for a unit vector $\hat{k}$

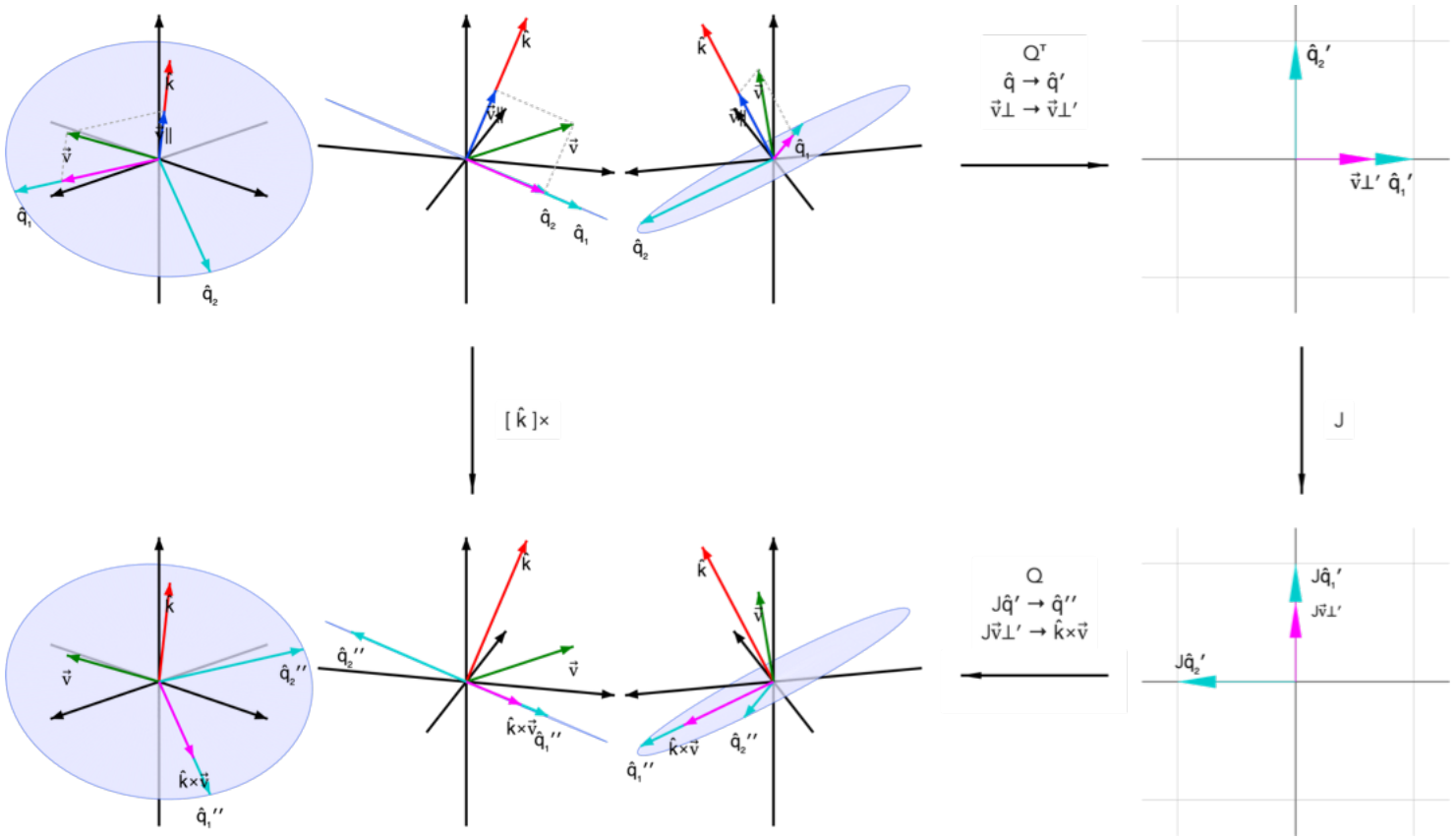
$$\hat{k} \times \vec{v} = \left([\,\hat{k}\,]\times\right)\vec{v}$$

is a composition of 3D and 2D transformations:

① project onto $\hat{k}^{\perp}$

② rotate by 90° in that plane with $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

③ return to $\mathbb{R}^3$



Let $Q = [\ \hat{q}_1 \mid \hat{q}_2\ ]$, where $\hat{q}_1$ and $\hat{q}_2$ form an orthonormal basis of $\hat{k}\perp$, chosen so that $\det[\ \hat{k} \mid \hat{q}_1 \mid \hat{q}_2\ ] > 0$.

$$[\ \hat{k}\ ]\times = Q\,J\,Q^{T} = \left[\begin{array}{c|c} \hat{q}_1 & \hat{q}_2 \end{array}\right]\left[\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right]\left[\begin{array}{c} \hat{q}_1^{\,T} \\ \hline \hat{q}_2^{\,T} \end{array}\right] = \left[\begin{array}{c|c} \hat{q}_2 & -\hat{q}_1 \end{array}\right]\left[\begin{array}{c} \hat{q}_1^{\,T} \\ \hline \hat{q}_2^{\,T} \end{array}\right] =$$

$$\hat{q}_2\,\hat{q}_1^{\,T} - \hat{q}_1\,\hat{q}_2^{\,T} = \left[\begin{array}{c|c|c} 0 & q_{12}\times q_{21} - q_{11}\times q_{22} & q_{12}\times q_{31} - q_{11}\times q_{32} \\ \hline -(q_{12}\times q_{21} - q_{11}\times q_{22}) & 0 & q_{22}\times q_{31} - q_{21}\times q_{32} \\ \hline -(q_{12}\times q_{31} - q_{11}\times q_{32}) & -(q_{22}\times q_{31} - q_{21}\times q_{32}) & 0 \end{array}\right]$$

↓

$[\ \hat{k}\ ]\times$ has a skew-symmetric form

The algebraic manipulations shown below show expected geometric actions of $[\ \hat{k}$

$]\times$

- $\left([\,\hat{k}\,]\times\right)\,\hat{k} = (\hat{q}_2\,\hat{q}_1^{\,T} - \hat{q}_1\,\hat{q}_2^{\,T})\,\hat{k} =$

$\hat{q}_2\,(\hat{q}_1^{\,T}\,\hat{k}) - \hat{q}_1\,(\hat{q}_2^{\,T}\,\hat{k}) = \vec{0}$

$\leftrightarrow$

maps $\hat{k}$ to $\vec{0}$

- $\left([\,\hat{k}\,]\times\right)\,\hat{q}_1 = (\hat{q}_2\,\hat{q}_1^{\,T} - \hat{q}_1\,\hat{q}_2^{\,T})\,\hat{q}_1 =$

$\hat{q}_2\,(\hat{q}_1^{\,T}\,\hat{q}_1) - \hat{q}_1\,(\hat{q}_2^{\,T}\,\hat{q}_1) =$

$\hat{q}_2 - 0\,\hat{q}_1 = \hat{q}_2$

$\leftrightarrow$

rotates $\hat{q}_1 \rightarrow \hat{q}_2$

- $\left([\,\hat{k}\,]\times\right)\,\hat{q}_2 = (\hat{q}_2\,\hat{q}_1^{\,T} - \hat{q}_1\,\hat{q}_2^{\,T})\,\hat{q}_2 =$

$\hat{q}_2\,(\hat{q}_1^{\,T}\,\hat{q}_2) - \hat{q}_1\,(\hat{q}_2^{\,T}\,\hat{q}_2) =$

$\hat{q}_2\,0 - \hat{q}_1 = -\,\hat{q}_1$

$\leftrightarrow$

rotates $\hat{q}_2 \rightarrow -\hat{q}_1$

The entries of $[\,\hat{k}\,]\times$ are independent of our choice of $\hat{q}_1$ and $\hat{q}_2$, and will be derived next.

Derivation through geometric action

As was shown above, $[\,\hat{k}\,]\times$ has a skew-symmetric form:

$$\hat{q}_2\,\hat{q}_1^{\intercal} - \hat{q}_1\,\hat{q}_2^{\intercal} = \begin{bmatrix} 0 & q_{12} \times q_{21} - q_{11} \times q_{22} & q_{12} \times q_{31} - q_{11} \times q_{32} \\ q_{22} \times q_{11} - q_{21} \times q_{12} & 0 & q_{22} \times q_{31} - q_{21} \times q_{32} \\ q_{32} \times q_{11} - q_{31} \times q_{12} & q_{32} \times q_{21} - q_{31} \times q_{22} & 0 \end{bmatrix}$$

The above matrix can be re-written as $\left([\,\hat{k}\,]\times\right) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

Geometric action of $[\,\hat{k}\,]\times$:

$$\left([\,\hat{k}\,]\times\right)\hat{k} = \vec{0}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

- $-a_3\,k_2 + a_2\,k_3 = 0$
- $a_3\,k_1 - a_1\,k_3 = 0$
- $-a_2\,k_1 + a_1\,k_2 = 0$

$$\downarrow$$

$a_i k_j = a_j k_i$ for every $i, j \in \{\,1, 2, 3\,\}$

Choose any r so that $k_r \neq 0$ and set $t = \dfrac{a_r}{k_r}$

$$\downarrow$$

$$a_i k_r = a_r k_i = t\, k_r\, k_i$$

$$\downarrow$$

$$a_i = t\, k_i$$

$$\downarrow$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = t \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

To find the scalar t, compare the three above-diagonal entries:

- $q_{21}q_{32} - q_{22}q_{31} = t\, k_1$
- $q_{12}q_{31} - q_{11}q_{32} = t\, k_2$
- $q_{11}q_{22} - q_{12}q_{21} = t\, k_3$

Now consider the positively oriented orthonormal basis

$$[\, \hat{\mathbf{k}} \mid \hat{\mathbf{q}}_1 \mid \hat{\mathbf{q}}_2 \,] = \left[\begin{array}{c|c|c} k_1 & q_{11} & q_{12} \\ \hline k_2 & q_{21} & q_{22} \\ \hline k_3 & q_{31} & q_{32} \end{array}\right] \qquad \det \left(\left[\begin{array}{c|c|c} k_1 & q_{11} & q_{12} \\ \hline k_2 & q_{21} & q_{22} \\ \hline k_3 & q_{31} & q_{32} \end{array}\right] \right) = 1$$

Expand this determinant along its first column:

$$
\begin{aligned}
&1 \\
&= k_1 (q_{21}q_{32} - q_{22}q_{31}) \\
&+ k_2 (q_{12}q_{31} - q_{11}q_{32}) \\
&+ k_3 (q_{11}q_{22} - q_{12}q_{21})
\end{aligned}
$$

Substitute the three entry-by-entry equalities:

$$
\begin{aligned}
&= k_1 (t\, k_1) + k_2 (t\, k_2) + k_3 (t\, k_3) \\
&= t (k_1^2 + k_2^2 + k_3^2)
\end{aligned}
$$

Because $\hat{k}$ is a unit vector,

$$k_1^2 + k_2^2 + k_3^2 = 1$$

$$\downarrow$$

$$t = 1$$

$$\downarrow$$

$$[\,\hat{k}\,]_\times = Q\,J\,Q^T = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

Null space by row reduction

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

↓

- $-a_3 k_2 + a_2 k_3 = 0$
- $a_3 k_1 - a_1 k_3 = 0$
- $-a_2 k_1 + a_1 k_2 = 0$

↓

Now rewrite it so that a_1, a_2, a_3 are the unknowns:
(this is where the three independent entries
of the 3×3 skew-symmetric matrix can be identified as one vector of unknowns)

$$\begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

case ①: $k_3 \neq 0$

$$\left[\begin{array}{ccc|c} k_3 & 0 & -k_1 & 0 \\ 0 & -k_3 & k_2 & 0 \\ -k_2 & k_1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} k_3 & 0 & -k_1 & 0 \\ 0 & -k_3 & k_2 & 0 \\ 0 & k_1 k_3 & -k_1 k_2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} k_3 & 0 & -k_1 & 0 \\ 0 & -k_3 & k_2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{k_1}{k_3} & 0 \\ 0 & 1 & -\frac{k_2}{k_3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

↓

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = t \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

Case ②: $k_3 = 0$, $k_2 \neq 0$

$$\left[\begin{array}{ccc|c} 0 & 0 & -k_1 & 0 \\ 0 & 0 & k_2 & 0 \\ -k_2 & k_1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -k_2 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & -k_1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -k_2 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{k_1}{k_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

↓

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = t \begin{bmatrix} k_1 \\ k_2 \\ 0 \end{bmatrix} = t \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

Case ③: $k_3 = 0$ and $k_2 = 0$

$$\left[\begin{array}{ccc|c} 0 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & k_1 & 0 & 0 \\ 0 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

↓

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = t \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

↓

$$\begin{bmatrix} 0 & q_{12} \times q_{21} - q_{11} \times q_{22} & q_{12} \times q_{31} - q_{11} \times q_{32} \\ q_{22} \times q_{11} - q_{21} \times q_{12} & 0 & q_{22} \times q_{31} - q_{21} \times q_{32} \\ q_{32} \times q_{11} - q_{31} \times q_{12} & q_{32} \times q_{21} - q_{31} \times q_{22} & 0 \end{bmatrix} = t \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

Derivation through single rotation inside $\hat{k}\ \vec{e}_1$ plane

We now derive the entries of the cross product matrix $[\ \hat{k}\ ]\times$
for a unit vector $\hat{k} = [\ k_1 \mid k_2 \mid k_3\ ]^T$

We change basis so that $\hat{k}$ becomes $\vec{e}_1$.

This simplest cross product matrix $[\ \vec{e}_1\ ]\times$ works as follows:
① maps the $\vec{e}_1$ direction to $\vec{0}$

$$\left([\ \vec{e}_1\ ]\times\right)\ \vec{e}_1 = \vec{0}$$

② rotates the $\vec{e}_2\ \vec{e}_3$ plane by 90°:

- $\left([\ \vec{e}_1\ ]\times\right)\ \vec{e}_2 = \vec{e}_3$
- $\left([\ \vec{e}_1\ ]\times\right)\ \vec{e}_3 = -\vec{e}_2$

↓

$$[\,\vec{e}_1\,]\times = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & -1 \\ \hline 0 & 1 & 0 \end{array}\right]$$

We need an orthogonal matrix U satisfying

- $U\,\hat{k} = \vec{e}_1$
- $U^{-1} = U^T$
- det(U) = 1, so that U preserves orientation

Once such a U is chosen, the cross product matrix for $\hat{k}$ is obtained by

- changing basis with U
- applying $[\,\vec{e}_1\,]\times$
- changing back with U^T:

$$[\,\hat{k}\,]\times = U^T\,[\,\vec{e}_1\,]\times\,U$$

We construct U as a single rotation inside the $\hat{k}\,\vec{e}_1$ plane

If $\hat{k} \neq \vec{e}_1$ and $\hat{k} \neq -\vec{e}_1$, $k_2^2 + k_3^2 \neq 0$

$$\text{let } \rho = \sqrt{k_2^2 + k_3^2} = \sqrt{1 - k_1^2}$$

We define the $\hat{k}\,\vec{e}_1$ plane by two orthonormal vectors:

- $\hat{u}_1 = \vec{e}_1$
- $\hat{u}_2$ is obtained by subtracting the projection of $\hat{k}$ onto $\vec{e}_1$ and normalizing:

$$\hat{u}_2 = \frac{\hat{k} - (\vec{e}_1^{\,\intercal}\,\hat{k})\,\vec{e}_1}{\rho} = \frac{\hat{k} - k_1\,\vec{e}_1}{\rho} = \begin{bmatrix} 0 \\ \frac{k_2}{\rho} \\ \frac{k_3}{\rho} \end{bmatrix}$$

Define B as the orthonormal basis of the rotation plane:

$$B = [\,\vec{e}_1 \mid \hat{u}_2\,] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \frac{k_2}{\rho} \\ \hline 0 & \frac{k_3}{\rho} \end{array}\right]$$

The angle θ between $\vec{e}_1$ and $\hat{k}$ can be found directly in $\mathbb{R}^3$:

$$\cos(\theta) = \frac{\vec{e}_1^{\,\intercal}\,\hat{k}}{|\,\vec{e}_1\,|\,|\,\hat{k}\,|} = k_1$$

$$\downarrow$$

$$\sin(\theta) = \sqrt{1 - k_1^2} = \rho$$

$\hat{k}$ lies at angle θ from $\vec{e}_1$, so to align $\hat{k}$ with $\vec{e}_1$ we rotate by $-\theta$:

↓

$$C = \left[\begin{array}{c|c} \cos\theta & \sin\theta \\ \hline -\sin\theta & \cos\theta \end{array}\right] = \left[\begin{array}{c|c} k_1 & \rho \\ \hline -\rho & k_1 \end{array}\right]$$

Matrix of projection onto the plane described by B is
$B\,(B^T\,B)^{-1}\,B^T$

$B^T\,B = I_2$
↓
$B\,(B^T\,B)^{-1}\,B^T = B\,B^T$
↓
Projection onto the direction perpendicular to B:
$P = I - B\,B^T$

U performs the following geometric actions:
① leaves all vectors perpendicular to B unchanged
② rotates all vectors inside the plane by $-\theta$ or $B\,C\,B^T$

Any vector $\vec{v}$ can be decomposed as
$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$,
where
• $\vec{v}_{\parallel}$ is inside the plane described by B:
$\vec{v}_{\parallel} = B\,B^T\,\vec{v}$
• $\vec{v}_{\perp}$ is perpendicular to that plane:
• $\vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel} = (I - B\,B^T)\,\vec{v}$

• $\vec{v}'_{\parallel} = B\,C\,B^T\,\vec{v}_{\parallel}$

• $\vec{v}'_{\perp} = \vec{v}_{\perp} = (I - B\,B^T)\,\vec{v}$

↓

$\vec{v}' = U\,\vec{v} = (\vec{v}'_{\parallel} + \vec{v}'_{\perp}) = B\,C\,B^T\,\vec{v} + (I - B\,B^T)\,\vec{v}$

$$B\,C = \begin{bmatrix} k_1 & \rho \\ -k_2 & \frac{k_1 k_2}{\rho} \\ -k_3 & \frac{k_1 k_3}{\rho} \end{bmatrix}$$

$$B\,C\,B^T = \begin{bmatrix} k_1 & k_2 & k_3 \\ -k_2 & \frac{k_1 k_2^2}{\rho^2} & \frac{k_1 k_2 k_3}{\rho^2} \\ -k_3 & \frac{k_1 k_2 k_3}{\rho^2} & \frac{k_1 k_3^2}{\rho^2} \end{bmatrix}$$

$$B\,B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{k_2^2}{\rho^2} & \frac{k_2 k_3}{\rho^2} \\ 0 & \frac{k_2 k_3}{\rho^2} & \frac{k_3^2}{\rho^2} \end{bmatrix}$$

$$I - B\,B^{T} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 - \frac{k_2^2}{\rho^2} & \frac{-k_2 k_3}{\rho^2} \\ 0 & \frac{-k_2 k_3}{\rho^2} & 1 - \frac{k_3^2}{\rho^2} \end{bmatrix}$$

$$U = B\,C\,B^{T} + (I - B\,B^{T}) =$$

$$\begin{bmatrix} k_1 & k_2 & k_3 \\ -k_2 & \frac{k_1 k_2^2}{\rho^2} & \frac{k_1 k_2 k_3}{\rho^2} \\ -k_3 & \frac{k_1 k_2 k_3}{\rho^2} & \frac{k_1 k_3^2}{\rho^2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 - \frac{k_2^2}{\rho^2} & \frac{-k_2 k_3}{\rho^2} \\ 0 & \frac{-k_2 k_3}{\rho^2} & 1 - \frac{k_3^2}{\rho^2} \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_3 \\ -k_2 & 1 + \frac{(k_1-1)k_2^2}{\rho^2} & \frac{(k_1-1)k_2 k_3}{\rho^2} \\ -k_3 & \frac{(k_1-1)k_2 k_3}{\rho^2} & 1 + \frac{(k_1-1)k_3^2}{\rho^2} \end{bmatrix}$$

Since

$$\rho^2 = 1 - k_1^2 = (1-k_1)(1+k_1),$$

$$\frac{k_1-1}{\rho^2} = \frac{-1}{1+k_1}$$

$$\downarrow$$

$$U = \begin{bmatrix} k_1 & k_2 & k_3 \\ -k_2 & 1 - \frac{k_2^2}{1+k_1} & \frac{-k_2 k_3}{1+k_1} \\ -k_3 & \frac{-k_2 k_3}{1+k_1} & 1 - \frac{k_3^2}{1+k_1} \end{bmatrix}$$

The first row of U is $\hat{k}^T$, so U $\hat{k} = \vec{e}_1$.

Also,
- U is orthogonal
- det(U) = 1
- $U^{-1} = U^T$

This construction used only
- a dot product
- an orthogonal projection
- normalization
- an embedded 2D rotation

No cross product matrix was used to construct U.

Now compute $[\vec{e}_1]\times$ U =

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \\ -k_2 & 1 - \frac{k_2^2}{1+k_1} & \frac{-k_2k_3}{1+k_1} \\ -k_3 & \frac{-k_2k_3}{1+k_1} & 1 - \frac{k_3^2}{1+k_1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ k_3 & \frac{k_2k_3}{1+k_1} & -1 + \frac{k_3^2}{1+k_1} \\ -k_2 & 1 - \frac{k_2^2}{1+k_1} & \frac{-k_2k_3}{1+k_1} \end{bmatrix}$$

Now change back by multiplying by U^T on the left.
The unsimplified entries are shown before the final cancellation:
↓

$$U^T\left([\,\vec{e}_1\,]\times U\right) = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -1+\dfrac{k_2^2+k_3^2}{1+k_1} \\ -k_2 & 1-\dfrac{k_2^2+k_3^2}{1+k_1} & 0 \end{bmatrix}$$

Since $\hat{k}$ is a unit vector,

$$k_1^2 + k_2^2 + k_3^2 = 1$$

Therefore,

$$k_2^2 + k_3^2 = 1 - k_1^2 = (1-k_1)(1+k_1)$$

So

$$-1+\frac{k_2^2+k_3^2}{1+k_1} = -k_1$$

$$1-\frac{k_2^2+k_3^2}{1+k_1} = k_1$$

$$\downarrow$$

$$[\,\hat{k}\,]\times = U^T[\,\vec{e}_1\,]\times U = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

For the two parallel cases:

- if $\hat{k} = \vec{e}_1$, choose U = I
- if $\hat{k} = -\vec{e}_1$, choose a 180° rotation in the $\vec{e}_1\ \vec{e}_2$ plane

Both cases give the same final matrix directly.

Derivation through change of basis, Householder reflection

We now derive the entries of the cross product matrix $[\ \hat{k}\ ]\times$ for a unit vector $\hat{k} = [\ k_1\ |\ k_2\ |\ k_3\ ]^T$

We change basis so that $\hat{k}$ becomes $\vec{e}_1$.

This simplest cross product matrix $[\ \vec{e}_1\ ]\times$ works as follows:

① maps the $\vec{e}_1$ direction to $\vec{0}$

$$\left([\ \vec{e}_1\ ]\times\right)\ \vec{e}_1 = \vec{0}$$

② rotates the $\vec{e}_2\ \vec{e}_3$ plane by 90°:

- $\left([\ \vec{e}_1\ ]\times\right)\ \vec{e}_2 = \vec{e}_3$
- $\left([\ \vec{e}_1\ ]\times\right)\ \vec{e}_3 = -\vec{e}_2$

↓

$$[\ \vec{e}_1\ ]\times = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & -1 \\ \hline 0 & 1 & 0 \end{array}\right]$$

Note that any reflection reverses orientation

↓

the 90° rotation in the reflected plane turns the opposite way

↓

the middle matrix has a reverse sign

↓

$$[\,\hat{k}\,]_\times = H^T \left(-[\,\vec{e}_1\,]_\times\right) H$$

Householder reflections are also symmetric, $H^T = H$

↓

$$[\,\hat{k}\,]_\times = -H\,[\,\vec{e}_1\,]_\times\, H$$

Let

$$\vec{w} = \hat{k} - \vec{e}_1$$

The Householder reflection across the plane perpendicular to $\vec{w}$ is

$$H = I - 2\,\frac{\vec{w}\,\vec{w}^T}{\vec{w}^T\,\vec{w}}$$

Note that choosing $\vec{w} = \vec{e}_1 - \hat{k}$ gives the same H since

- $(-\vec{w})^T(-\vec{w}) = \vec{w}^T\,\vec{w}$
- $(-\vec{w})(-\vec{w})^T = \vec{w}\,\vec{w}^T$

Next, will compute $\vec{w}^T\,\vec{w}$:

$$\vec{w}^T\,\vec{w} = (\hat{k} - \vec{e}_1)^T\,(\hat{k} - \vec{e}_1) =$$

$$(\hat{k}^T - \vec{e}_1{}^T)\,(\hat{k} - \vec{e}_1) =$$

$$\hat{k}^T\,\hat{k} - \vec{e}_1{}^T\,\hat{k} - \hat{k}^T\,\vec{e}_1 + \vec{e}_1{}^T\,\vec{e}_1 =$$

$$1 - k_1 - k_1 + 1 = 2\,(1 - k_1)$$

↓

$$H = I - \frac{(\hat{k} - \vec{e}_1)(\hat{k} - \vec{e}_1)^T}{1 - k_1} = \begin{bmatrix} k_1 & k_2 & k_3 \\ k_2 & 1 - \frac{k_2^2}{1-k_1} & \frac{-k_2 k_3}{1-k_1} \\ k_3 & \frac{-k_2 k_3}{1-k_1} & 1 - \frac{k_3^2}{1-k_1} \end{bmatrix}$$

Now compute $[\vec{e}_1]\times H$:

$$[\vec{e}_1]\times H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \\ k_2 & 1 - \frac{k_2^2}{1-k_1} & \frac{-k_2 k_3}{1-k_1} \\ k_3 & \frac{-k_2 k_3}{1-k_1} & 1 - \frac{k_3^2}{1-k_1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -k_3 & \frac{k_2 k_3}{1-k_1} & -1 + \frac{k_3^2}{1-k_1} \\ k_2 & 1 - \frac{k_2^2}{1-k_1} & \frac{-k_2 k_3}{1-k_1} \end{bmatrix}$$

Now multiply by $-H$ on the left.
The unsimplified entries are shown before the final cancellation:

$$-H^T([\vec{e}_1]\times H) = -H([\vec{e}_1]\times H) = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & 1 - \frac{k_2^2 + k_3^2}{1-k_1} \\ -k_2 & -1 + \frac{k_2^2 + k_3^2}{1-k_1} & 0 \end{bmatrix}$$

Since $\hat{k}$ is a unit vector,

$$k_1^2 + k_2^2 + k_3^2 = 1$$

Therefore,

$$k_2^2 + k_3^2 = 1 - k_1^2 = (1-k_1)(1+k_1)$$

So

$$1 - \frac{k_2^2+k_3^2}{1-k_1} = -k_1$$

$$-1 + \frac{k_2^2+k_3^2}{1-k_1} = k_1$$

$$\downarrow$$

$$[\hat{k}]_\times = -H^T [\vec{e}_1]_\times H = -H [\vec{e}_1]_\times H = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

Derivation through change of basis, Givens rotation

We now derive the entries of the cross product matrix $[\hat{k}]_\times$ for a unit vector $\hat{k} = [k_1 \mid k_2 \mid k_3]^T$

We change basis so that $\hat{k}$ becomes $\vec{e}_1$.

This simplest cross product matrix $[\vec{e}_1]\times$ works as follows:

① maps the $\vec{e}_1$ direction to $\vec{0}$

$$\left([\vec{e}_1]\times\right)\vec{e}_1 = \vec{0}$$

② rotates the $\vec{e}_2$ $\vec{e}_3$ plane by 90°:

- $\left([\vec{e}_1]\times\right)\vec{e}_2 = \vec{e}_3$
- $\left([\vec{e}_1]\times\right)\vec{e}_3 = -\vec{e}_2$

↓

$$[\vec{e}_1]\times = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

We need an orthogonal matrix G satisfying

- $G\,\hat{k} = \vec{e}_1$
- $G^{-1} = G^{\intercal}$
- det(G) = 1, so that G preserves orientation

Once such a G is chosen, the cross product matrix for $\hat{k}$ is obtained by

- changing basis with G
- applying $[\vec{e}_1]\times$
- changing back with $G^{\intercal}$:

$$[\hat{k}]\times = G^{\intercal}\,[\vec{e}_1]\times\,G$$

Now construct G.

Let

$$\rho = \sqrt{k_1^2 + k_3^2}$$

If $\rho = 0$, then $k_1 = 0$ and $k_3 = 0$.
Since $\hat{k}$ is a unit vector, $k_2 = \pm 1$.
So $\hat{k}$ is either $\vec{e}_2$ or $-\vec{e}_2$.

These are coordinate-axis cases.
Cross product by $\vec{e}_2$ maps

- $\vec{e}_1$ to $-\vec{e}_3$
- $\vec{e}_2$ to $\vec{0}$
- $\vec{e}_3$ to $\vec{e}_1$

↓

$$[\vec{e}_2]_\times = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

For $-\vec{e}_2$, all signs reverse:

$$[-\vec{e}_2]_\times = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Both coordinate-axis cases agree with the final formula.

Now assume $\rho \neq 0$.
In this generic case, construct G as a product of two Givens rotations:
$G = G_2 G_1$

① Rotate in the $\vec{e}_1$ $\vec{e}_3$-plane to remove the k_3 component:

$$G_1 = \begin{bmatrix} \frac{k_1}{\rho} & 0 & \frac{k_3}{\rho} \\ 0 & 1 & 0 \\ \frac{-k_3}{\rho} & 0 & \frac{k_1}{\rho} \end{bmatrix} \qquad G_1 \hat{k} = \begin{bmatrix} \rho \\ k_2 \\ 0 \end{bmatrix}$$

② Rotate in the $\vec{e}_1\,\vec{e}_2$-plane to remove the k_2 component:

$$G_2 = \begin{bmatrix} \rho & k_2 & 0 \\ -k_2 & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G = G_2\,G_1 = \begin{bmatrix} \rho & \frac{k_1}{\rho} & k_2 & \rho & \frac{k_3}{\rho} \\ -k_2 & \frac{k_1}{\rho} & \rho & -k_2 & \frac{k_3}{\rho} \\ \frac{-k_3}{\rho} & & 0 & \frac{k_1}{\rho} & \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_3 \\ \frac{-k_1 k_2}{\rho} & \rho & \frac{-k_2 k_3}{\rho} \\ \frac{-k_3}{\rho} & 0 & \frac{k_1}{\rho} \end{bmatrix}$$

$$[\,\vec{e}_1\,]\times G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \\ \frac{-k_1k_2}{\rho} & \rho & \frac{-k_2k_3}{\rho} \\ \frac{-k_3}{\rho} & 0 & \frac{k_1}{\rho} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1\left(\frac{-k_3}{\rho}\right) & 0 & -1\,\frac{k_1}{\rho} \\ \frac{-k_1k_2}{\rho} & \rho & \frac{-k_2k_3}{\rho} \end{bmatrix}$$

Now multiply by G^T on the left.
The unsimplified entries are shown first so the cancellation is visible:

$$G^T\left([\,\vec{e}_1\,]\times\right)G = \begin{bmatrix} k_1 & \frac{-k_1k_2}{\rho} & \frac{-k_3}{\rho} \\ k_2 & \rho & 0 \\ k_3 & \frac{-k_2k_3}{\rho} & \frac{k_1}{\rho} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -1\left(\frac{-k_3}{\rho}\right) & 0 & -1\,\frac{k_1}{\rho} \\ \frac{-k_1k_2}{\rho} & \rho & \frac{-k_2k_3}{\rho} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{-k_1k_2}{\rho}\left(-1\left(\frac{-k_3}{\rho}\right)\right) - \frac{k_3}{\rho}\left(\frac{-k_1k_2}{\rho}\right) & \frac{-k_3}{\rho}\,\rho & \frac{-k_1k_2}{\rho}\left(-1\,\frac{k_1}{\rho}\right) - \frac{k_3}{\rho}\left(\frac{-k_2k_3}{\rho}\right) \\ \rho\left(-1\left(\frac{-k_3}{\rho}\right)\right) & 0 & \rho\left(-1\,\frac{k_1}{\rho}\right) \\ \frac{-k_2k_3}{\rho}\left(-1\left(\frac{-k_3}{\rho}\right)\right) + \frac{k_1}{\rho}\left(\frac{-k_1k_2}{\rho}\right) & \frac{k_1}{\rho}\,\rho & \frac{-k_2k_3}{\rho}\left(-1\,\frac{k_1}{\rho}\right) + \frac{k_1}{\rho}\left(\frac{-k_2k_3}{\rho}\right) \end{bmatrix}$$

Using $\rho^2 = k_1^2 + k_3^2$ and $k_1^2 + k_2^2 + k_3^2 = 1$,
all ρ terms cancel
↓

The cross product matrix $[\hat{k}]_\times = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$

Consequences of the $[\hat{k}]_\times$ derivation

Ⓐ

Length of $\hat{k} \times \vec{v}$

The construction of $[\hat{k}]_\times$ first removes the component of $\vec{v}$ parallel to $\hat{k}$.
Only the perpendicular component $\vec{v}\perp$ remains.

Then $\vec{v}\perp$ is rotated by 90° in $\hat{k}\perp$ and returned to $\mathbb{R}^3$.
Both steps preserve length.

Therefore,
$|\hat{k} \times \vec{v}| = |\vec{v}\perp|$

If θ is the angle between $\hat{k}$ and $\vec{v}$,
$|\vec{v}\perp| = |\vec{v}| \sin(\theta)$

Ⓑ

Extension to a non-unit first vector

For a unit vector $\hat{k}$, the construction gives

$$\hat{k} \times \vec{v} = \left([\, \hat{k} \,]\times \right) \vec{v}$$

Now let $\vec{j}$ be a non-unit vector in the same direction:

$$\vec{j} = |\vec{j}\,| \, \hat{k}$$

The plane and the 90° rotation are determined by the direction $\hat{k}$.
The length $|\vec{j}\,|$ is included as part of the definition:

$$\vec{j} \times \vec{v} := |\vec{j}\,| \, (\hat{k} \times \vec{v})$$

Therefore,

$$\left([\, \vec{j} \,]\times \right) \vec{v} = |\vec{j}\,| \;\; \left([\, \hat{k} \,]\times \right) \vec{v}$$

Since this holds for every $\vec{v}$,

$$[\, \vec{j} \,]\times = |\vec{j}\,| \, [\, \hat{k} \,]\times$$

$$[\, \vec{j} \,]\times = |\vec{j}\,| \, [\, \hat{k} \,]\times = |\vec{j}\,| \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -j_3 & j_2 \\ j_3 & 0 & -j_1 \\ -j_2 & j_1 & 0 \end{bmatrix}$$

Because $\vec{j} = |\vec{j}\,| \, \hat{k}$,

$j_1 = |\vec{j}\,| \, k_1$, $j_2 = |\vec{j}\,| \, k_2$, $j_3 = |\vec{j}\,| \, k_3$.

So the non-unit matrix has the same skew-symmetric form,
with the entries of $\vec{j}$ itself.

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Determinant from the cross product

Let

$$\vec{n} = \vec{j} \times \vec{v}$$

- $\vec{n}$ is perpendicular to $\vec{j}$ and $\vec{v}$
- length of $\vec{n}$ was defined as

$$|\vec{n}| = |\vec{j}|\,|\vec{v}|\sin(\theta)$$

This is the area of the parallelogram spanned by $\vec{j}$ and $\vec{v}$.

So in the parallelepiped with base vectors $\vec{j}$, $\vec{v}$ and height vector $\vec{n}$:

- base area = $|\vec{n}|$
- height = $|\vec{n}|$

↓

$$\det\left([\,\vec{n}\,|\,\vec{j}\,|\,\vec{v}\,]\right) = \text{base area} \times \text{height} = |\vec{n}|^2$$

The determinant is positive because $\vec{n} = \vec{j} \times \vec{v}$ was chosen so that $(\vec{n}, \vec{j}, \vec{v})$ is positively oriented.