$\vec{k} \times \vec{v}$: definition of cross product in $\mathbb{R}^3$

Ⓐ

The cross product is a specifically 3D function that

- inputs two vectors in $\mathbb{R}^3$: $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

- outputs one vector in $\mathbb{R}^3$: $\begin{bmatrix} \\ \vec{k} \times \vec{v} \\ \\ \end{bmatrix}$

The output vector $\vec{k} \times \vec{v}$ is defined by direction and magnitude

① Direction

$\vec{k} \times \vec{v}$ is orthogonal to both input vectors:

$$(\vec{k} \times \vec{v}) \cdot \vec{k} = 0$$
$$(\vec{k} \times \vec{v}) \cdot \vec{v} = 0$$

If $\vec{k}$ and $\vec{v}$ are linearly independent,
there are two possible perpendicular directions

The direction is chosen so that,
when viewed from the tip of $\vec{k} \times \vec{v}$ toward the origin,
the shorter rotation from $\vec{k}$ to $\vec{v}$ appears counterclockwise

↓

- $(\vec{k} \times \vec{v}, \vec{k}, \vec{v})$ is a positively oriented triple in $\mathbb{R}^3$
- $\det\left([\,\vec{k} \times \vec{v} \mid \vec{k} \mid \vec{v}\,]\right) > 0$

If $\vec{k}$ and $\vec{v}$ are linearly dependent,
then $\vec{k} \times \vec{v} = \vec{0}$ and no direction is chosen

② Magnitude

The length of $\vec{k} \times \vec{v}$ is defined as
$| \vec{k} \times \vec{v} | = | \vec{k} | \, | \vec{v} | \sin(\theta)$

where θ is the angle between $\vec{k}$ and $\vec{v}$

This is the area of the parallelogram spanned by $\vec{k}$ and $\vec{v}$

Ⓑ
Connection to determinant concept:

If we build a 3D parallelepiped using
$\vec{k} \times \vec{v}$ as the height vector and $\vec{k}$, $\vec{v}$ as the base vectors with

- height = $| \vec{k} \times \vec{v} |$
- $(\vec{k} \times \vec{v}) \cdot \vec{k} = 0$
- $(\vec{k} \times \vec{v}) \cdot \vec{v} = 0$
- orientation such that $\det\left([\, \vec{k} \times \vec{v} \mid \vec{k} \mid \vec{v} \,]\right) > 0$

then

$$\det\left([\, \vec{k} \times \vec{v} \mid \vec{k} \mid \vec{v} \,]\right) = | \vec{k} \times \vec{v} |^2$$

Ⓒ
Cross product as composition of matrix operations:

For derivation simplicity, will assume that $| \vec{k} | = 1$
The next page shows the matrix $[\, \hat{k} \,]_\times$
as the operator that performs the following actions:

- removes the component of $\vec{v}$ parallel to $\hat{k}$
- expresses the remaining component of $\vec{v}$ in $\mathbb{R}^2$
- rotates that component by 90°
- re-embeds the result back in $\mathbb{R}^3$

$[\hat{k}]_\times$: cross product matrix for unit vector $\hat{k}$

Matrix $[\hat{k}]_\times$ as a composition of the following transformations:
① removes the projection of $\vec{v}$ onto $\hat{k}$, computed as $\hat{k}\,\hat{k}^T\,\vec{v}$
↔
projects $\vec{v}$ onto plane orthogonal to $\hat{k}$ as
$(I - \hat{k}\,\hat{k}^T)\,\vec{v}$
If we describe the plane by orthonormal matrix $Q = [\,\hat{q}_1 \mid \hat{q}_2\,]$,
$Q\,Q^T$ becomes the matrix of projection onto the plane
$Q\,(Q^T Q)^{-1}\,Q^T = Q\,Q^T$
with geometric actions identical to $I - \hat{k}\,\hat{k}^T$:
$I - \hat{k}\,\hat{k}^T = Q\,Q^T$

Image below shows
- $\hat{k}$ (red)
- $\vec{v}$ (green)
- $\vec{v}_{\|} = \hat{k}\,\hat{k}^T\,\vec{v}$ (blue)
- $\vec{v}_{\perp} = (I - \hat{k}\,\hat{k}^T)\,\vec{v}$ (magenta)
- $\hat{q}_1$ (cyan), can be computed as $\dfrac{\vec{v}_{\perp}}{|\,\vec{v}_{\perp}\,|}$

- $\hat{q}_2$ (cyan), computed from $\begin{bmatrix} \hat{q}_1 \\ \hat{k} \end{bmatrix} \vec{q} = \vec{0}$

the system has

‣ 2 linearly independent rows $\hat{q}_1$ & $\hat{k}$

‣ 3 variables

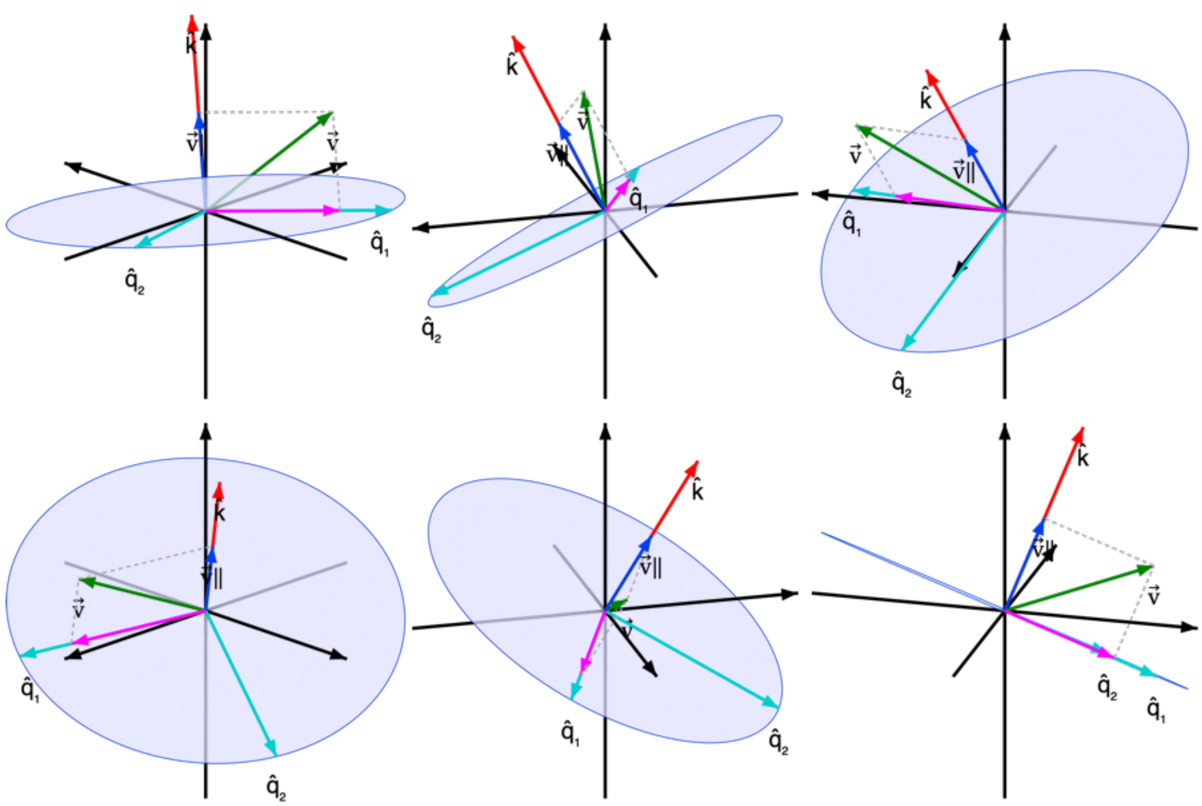
↓

‣ solution set is a line in $\mathbb{R}^3$

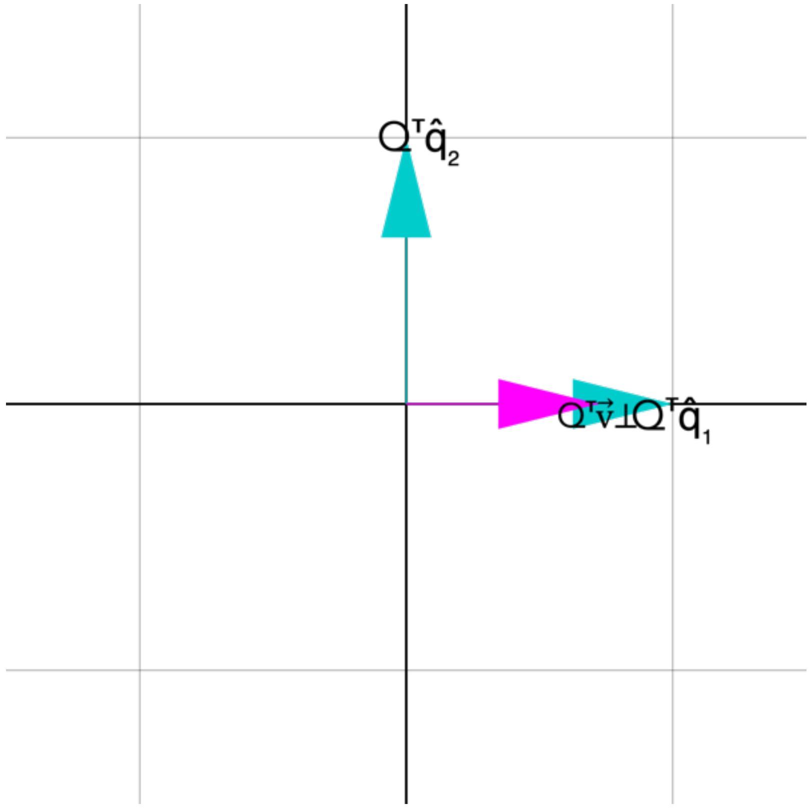
‣ $\vec{q}$ sign is chosen so that $\det\left([\,\hat{q}_1 \mid \vec{q} \mid \hat{k}\,]\right) > 0$

‣ amplitude is chosen as $\hat{q}_2 = \dfrac{\vec{q}}{|\vec{q}|}$

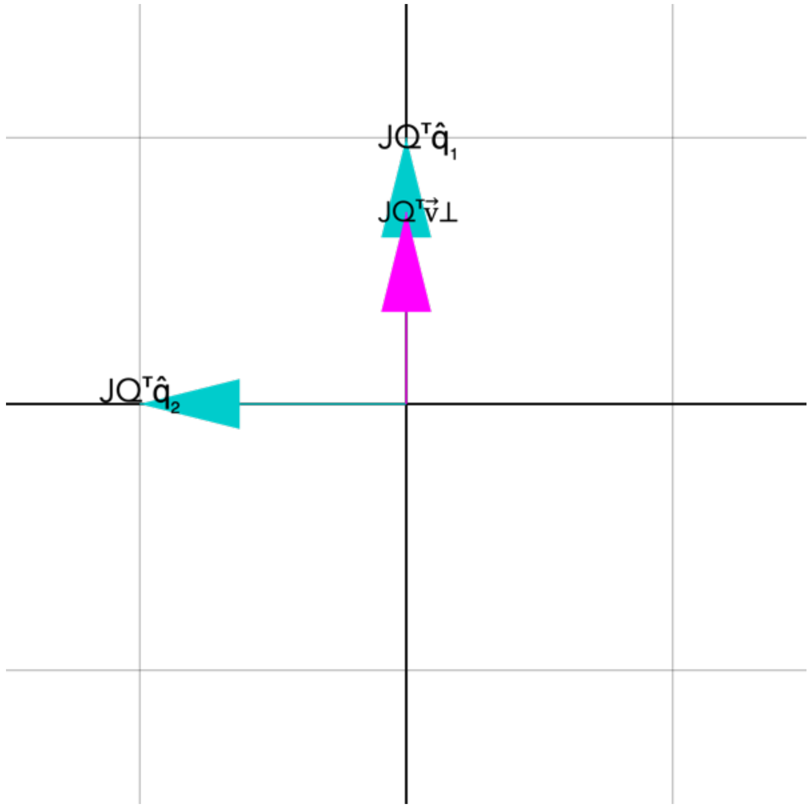
This unique determination of $\hat{q}_2$ from $\hat{q}_1$ and $\hat{k}$ is specific to $\mathbb{R}^3$



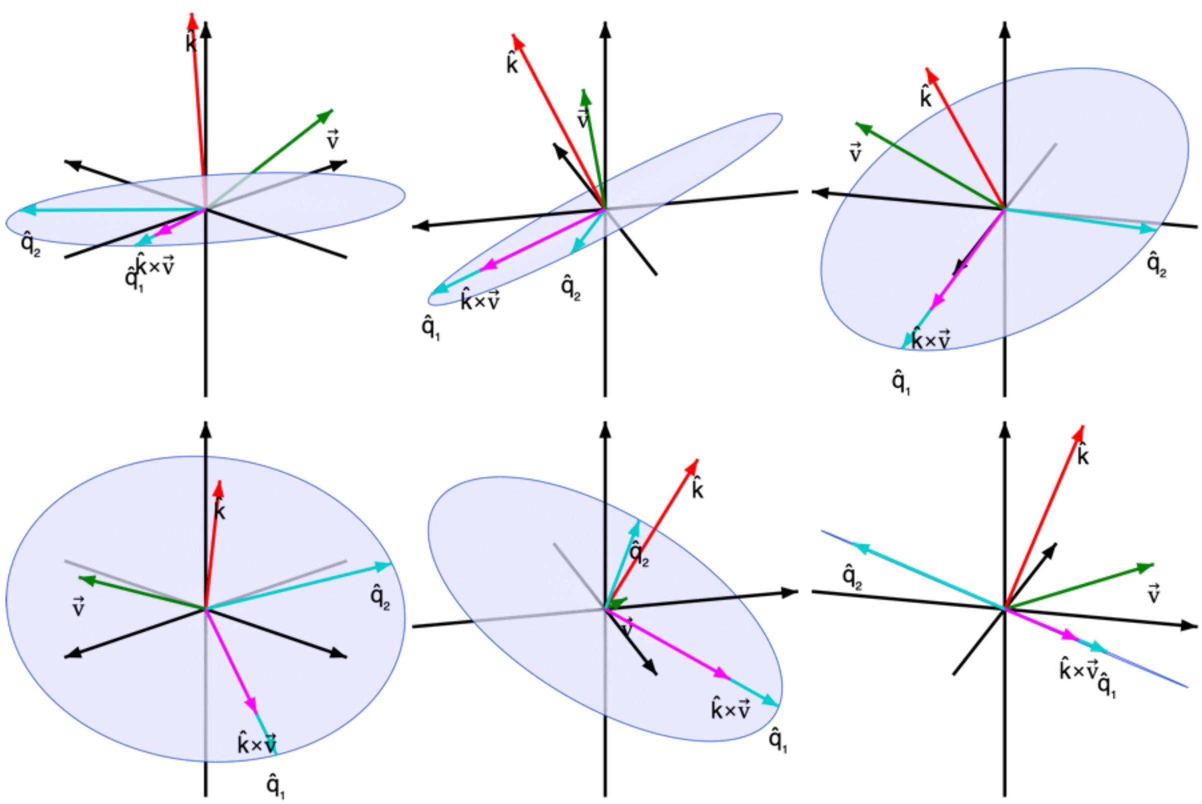
② expresses vectors in the orthogonal plane in $\mathbb{R}^2$ coordinates: $Q^T = \begin{bmatrix} \hat{q}_1^T \\ \hat{q}_2^T \end{bmatrix}$



③ applies the 90° rotation: $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



④ lifts back to $\mathbb{R}^3$: $Q = \left[\ \hat{q}_1 \ \middle|\ \hat{q}_2\ \right]$



$$(I - \hat{k}\,\hat{k}^T) = Q\,Q^T$$

$$\downarrow$$

$$[\,\hat{k}\,]\times = Q\,J\,Q^T\,(I - \hat{k}\,\hat{k}^T) =$$

$$Q\,J\,(Q^T\,Q)\,Q^T = Q\,J\,Q^T$$

$$Q\,J = \left[\begin{array}{c|c} \hat{q}_1 & \hat{q}_2 \end{array}\right]\left[\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right] = \left[\begin{array}{c|c} \hat{q}_2 & -\hat{q}_1 \end{array}\right]$$

$$[\,\hat{k}\,]\times = (Q\,J)\,Q^T = \left[\begin{array}{c|c} \hat{q}_2 & -\hat{q}_1 \end{array}\right]\left[\begin{array}{c} \hat{q}_1^{\,T} \\ \hat{q}_2^{\,T} \end{array}\right] = \hat{q}_2\,\hat{q}_1^{\,T} - \hat{q}_1\,\hat{q}_2^{\,T}$$

The algebraic manipulations shown below confirm expected geometric actions of $[\,\hat{k}\,]\times$

- $[\,\hat{k}\,]\times\,\hat{k} = (\hat{q}_2\,\hat{q}_1^{\,T} - \hat{q}_1\,\hat{q}_2^{\,T})\,\hat{k} =$
  $\hat{q}_2\,(\hat{q}_1^{\,T}\,\hat{k}) - \hat{q}_1\,(\hat{q}_2^{\,T}\,\hat{k}) = \vec{0}$
  $\leftrightarrow$
  maps $\hat{k}$ to $\vec{0}$

- $[\,\hat{k}\,]\times\,\hat{q}_1 = (\hat{q}_2\,\hat{q}_1^{\,T} - \hat{q}_1\,\hat{q}_2^{\,T})\,\hat{q}_1 =$
  $\hat{q}_2\,(\hat{q}_1^{\,T}\,\hat{q}_1) - \hat{q}_1\,(\hat{q}_2^{\,T}\,\hat{q}_1) =$
  $\hat{q}_2 - 0\,\hat{q}_1 = \hat{q}_2$
  $\leftrightarrow$
  rotates $\hat{q}_1 \rightarrow \hat{q}_2$

- $[\,\hat{k}\,]\times\,\hat{q}_2 = (\hat{q}_2\,\hat{q}_1^{\,T} - \hat{q}_1\,\hat{q}_2^{\,T})\,\hat{q}_2 =$
  $\hat{q}_2\,(\hat{q}_1^{\,T}\,\hat{q}_2) - \hat{q}_1\,(\hat{q}_2^{\,T}\,\hat{q}_2) =$
  $\hat{q}_2\,0 - \hat{q}_1 = -\,\hat{q}_1$
  $\leftrightarrow$
  rotates $\hat{q}_2 \rightarrow -\hat{q}_1$

The above identities uniquely describe
how $[\,\hat{k}\,]\times$ transforms the 3 spanning vectors

Note the particular form of the matrix, called skew-symmetric

where $[\,\hat{k}\,]\times_{i,j} = -[\,\hat{k}\,]\times_{j,i}$ for $i \neq j$

$$\hat{q}_2\,\hat{q}_1^{T} - \hat{q}_1\,\hat{q}_2^{T} = \begin{bmatrix} 0 & q_{12} \times q_{21} - q_{11} \times q_{22} & q_{12} \times q_{31} - q_{11} \times q_{32} \\ q_{22} \times q_{11} - q_{21} \times q_{12} & 0 & q_{22} \times q_{31} - q_{21} \times q_{32} \\ q_{32} \times q_{11} - q_{31} \times q_{12} & q_{32} \times q_{21} - q_{31} \times q_{22} & 0 \end{bmatrix}$$

On the next page we will derive
the unique matrix that satisfies all the geometric properties
independent of choice of $\hat{q}_1$ & $\hat{q}_2$

$$[\,\hat{k}\,]\times = Q\,J\,Q^{T} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

Computing entries of $[\,\hat{k}\,]\times$

As was shown above, $[\,\hat{k}\,]\times$ has a skew-symmetric form:

$$\hat{q}_2 \hat{q}_1^T - \hat{q}_1 \hat{q}_2^T = \begin{bmatrix} 0 & q_{12} \times q_{21} - q_{11} \times q_{22} & q_{12} \times q_{31} - q_{11} \times q_{32} \\ q_{22} \times q_{11} - q_{21} \times q_{12} & 0 & q_{22} \times q_{31} - q_{21} \times q_{32} \\ q_{32} \times q_{11} - q_{31} \times q_{12} & q_{32} \times q_{21} - q_{31} \times q_{22} & 0 \end{bmatrix}$$

The above matrix can be re-written as $\left([\,\hat{k}\,]\times\right) = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$

Recall geometric actions of $[\,\hat{k}\,]\times$:

- maps $\hat{k}$ to $\vec{0}$
- rotates $\hat{q}_1 \rightarrow \hat{q}_2$
- rotates $\hat{q}_2 \rightarrow -\hat{q}_1$

↓

col$\left([\,\hat{k}\,]\times\right)$ is a plane in $\mathbb{R}^3{}_1$

↓

3×3 matrix $[\,\hat{k}\,]\times$ has rank = 2

↓

null$\left([\,\hat{k}\,]\times\right)$ is a line in $\mathbb{R}^3$

↓

any vector in null$\left([\,\hat{k}\,]\times\right)$ is a scalar multiple of $\hat{k}$

From algebraic properties of skew-symmetric $\left([\,\hat{k}\,]\times\right)$, we can infer that

$$\text{null}\left([\hat{k}]\times\right) \text{ is spanned by a vector } \vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

↓

same 1-dimensional space is spanned by $\hat{k}$ & $\vec{a}$

↓

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (d\,\hat{k}) \text{ where d is a scalar}$$

This a, b, c parametrization is another $\mathbb{R}^3$-specific step:
only in 3D does a skew-symmetric matrix correspond to a single vector

Next, we need to find the unique d that satisfies

- $\left([\hat{k}]\times\right) \hat{q}_1 = \hat{q}_2$
- $\left([\hat{k}]\times\right) \hat{q}_2 = -\hat{q}_1$

We will consider a simple case of $\mathbb{R}^3$ where

- $\vec{e}_3 = \hat{k}$
- $\vec{e}_1 = \hat{q}_1$
- $\vec{e}_2 = \hat{q}_2$

In this case,

- $d \left([\hat{k}]\times\right) \vec{e}_1 = \vec{e}_2$

- $$d \left([\hat{k}]_\times \right) \vec{e}_1 = d \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$d = 1$ is the only scalar that satisfies the above

↓

$$[\hat{k}]_\times = Q J Q^T = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

$[\vec{j}\,]_\times$: cross product matrix for non-unit vector $\vec{j}$

Cross product is equivalent to left-multiplication by $[\vec{j}\,]_\times$

$$\vec{j} \times \vec{v} = [\vec{j}\,]_\times \vec{v}$$

For a unit vector $\hat{k}$: $[\hat{k}]_\times = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$

Now let $\vec{j}$ be a non-unit vector in the same direction:

$$\vec{j} = |\vec{j}|\,\hat{k}$$

The cross product length definition gives

$$|\vec{j} \times \vec{v}| = |\vec{j}|\,|\vec{v}|\sin(\theta)$$

So compared with $\hat{k} \times \vec{v}$,
the direction is the same,
but the length is multiplied by $|\vec{j}|$:

$$\vec{j} \times \vec{v} = |\vec{j}|\,(\hat{k} \times \vec{v})$$

Therefore,

$$[\vec{j}\,]_\times \vec{v} = |\vec{j}|\,[\,\hat{k}\,]_\times \vec{v}$$

so

$$[\vec{j}\,]_\times = |\vec{j}|\,[\,\hat{k}\,]_\times$$

Since $\vec{j} = |\vec{j}|\,\hat{k}$,
$[\vec{j}\,]_\times$ is composed of the same entries of $\vec{j}$:

$$[\vec{j}\,]_\times = \begin{bmatrix} 0 & -j_3 & j_2 \\ j_3 & 0 & -j_1 \\ -j_2 & j_1 & 0 \end{bmatrix}$$

Length of $[\hat{k}]_\times \vec{v}$

For unit $\hat{k}$,

$$[\hat{k}]_\times = Q\, J\, Q^T$$

$$\downarrow$$

$$\hat{k} \times \vec{v} = [\hat{k}]_\times \vec{v} = Q\, J\, Q^T \vec{v}$$

$$\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$$

$$\downarrow$$

$$Q\, J\, Q^T \vec{v} = (Q\, J\, Q^T)(\vec{v}_\parallel + \vec{v}_\perp) =$$

$$Q\, J\, (Q^T \vec{v}_\parallel + Q^T \vec{v}_\perp) =$$

$$Q\, J\, (\vec{0} + Q^T \vec{v}_\perp) = Q\, J\, Q^T \vec{v}_\perp$$

ⓐ

- $Q^T \vec{v}_\perp = \begin{bmatrix} \hat{q}_1^T \\ \hat{q}_2^T \end{bmatrix} \vec{v}_\perp = \begin{bmatrix} \hat{q}_1^T \vec{v}_\perp \\ \hat{q}_2^T \vec{v}_\perp \end{bmatrix} = \begin{bmatrix} |\vec{v}_\perp| \\ 0 \end{bmatrix}$

- $\hat{q}_1$ is chosen as colinear with $\vec{v}_\perp$
- $|\hat{q}_1| = 1$ and $\hat{q}_2 \perp \vec{v}_\perp$

$$\downarrow$$

$$|Q^T \vec{v}_\perp| = |\vec{v}_\perp|$$

ⓑ

J is a rotation matrix with orthonormal columns

$$\downarrow$$

$$|J\, Q^T \vec{v}_\perp| = |Q^T \vec{v}_\perp|$$

ⓒ

if $\vec{w} = J\, Q^T\, \vec{v}_\perp$ (2×1)

$$\bullet\ Q\,\vec{w} = \left[\ \hat{q}_1 \ \middle|\ \hat{q}_2\ \right]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \left[\ \hat{q}_1 w_1 + \hat{q}_2 w_2\ \right]$$

$$\downarrow$$

$$|\, Q\,\vec{w}\,| = |\,\vec{w}\,|$$

Combining ⓐ, ⓑ, and ⓒ:

$$|\,\hat{k} \times \vec{v}\,| = |\, Q\, J\, Q^T\, \vec{v}\,|$$
$$= |\, Q\, J\, Q^T\, \vec{v}_\perp\,| = |\, J\, Q^T\, \vec{v}_\perp\,|$$
$$= |\, Q^T\, \vec{v}_\perp\,| = |\,\vec{v}_\perp\,|$$

Therefore, for unit $\hat{k}$,

$$|\,\hat{k} \times \vec{v}\,| = |\,\vec{v}_\perp\,|$$

If θ is the angle between $\hat{k}$ and $\vec{v}$,

$$|\,\vec{v}_\perp\,| = |\,\vec{v}\,| \sin(\theta)$$
$$\downarrow$$
$$|\,\hat{k} \times \vec{v}\,| = |\,\hat{k}\,|\,|\,\vec{v}\,| \sin(\theta) = |\,\vec{v}\,| \sin(\theta)$$

which connects this to cross product definition

Connection with cross-product determinant mnemonic

The commonly taught memorization method is not an ordinary matrix or determinant:

- the first row contains standard basis vectors
- the other rows contain scalar components of $\vec{a}$ and $\vec{b}$

$$\vec{a} \times \vec{b} = \text{'det'}\left(\begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\right)$$

will denote

$$C1 = \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \quad C2 = \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} \quad C3 = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

$\vec{a} \times \vec{b}$ is read as symbolic 'cofactor expansion':
$\vec{a} \times \vec{b} = \hat{e}_1 \det(C1) - \hat{e}_2 \det(C2) + \hat{e}_3 \det(C3) =$

$$\begin{bmatrix} \det(C1) \\ -\det(C2) \\ \det(C3) \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_1b_3 - a_3b_1 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

$$\text{Since } [\,\vec{a}\,]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -a_3 b_2 + a_2 b_3 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix}$$