Cramer's rule

If A is square and $|A| \neq 0$, Cramer's rule computes the solution $\vec{x}$ of $A\vec{x} = \vec{b}$ from determinants

① Suppose we have matrix

$$X_k = [\, \vec{e}_1 \mid \dots \mid \vec{x} \mid \dots \mid \vec{e}_n \,]$$

where $\vec{x} = [\, x_1 \mid \dots \mid x_n \,]^T$ is the k-th column of X_k

example of 4×4 matrix with k = 3 is shown below:

$$X_k = \begin{bmatrix} 1 & 0 & x_1 & 0 \\ 0 & 1 & x_2 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & x_4 & 1 \end{bmatrix}$$

X_k can be obtained from identity matrix I_n by the following steps:

- for each $i \neq k$, adding x_i times row k to row i
- scaling row k by x_k

- row additions do not change determinant
- row scaling changes determinant by scaling factor

↓

$$|X_k| = x_k$$

② Next, compute the product

$A_k = A X_k$ for $A = [\, \vec{a}_1 \mid \dots \mid \vec{a}_k \mid \dots \mid \vec{a}_n \,]$

$$A X_k = [\, A\vec{e}_1 \mid \dots \mid A\vec{x} \mid \dots \mid A\vec{e}_n \,] =$$

$$A X_k = [\, \vec{a}_1 \mid \dots \mid A\vec{x} \mid \dots \mid \vec{a}_n \,] =$$

$$A X_k = [\, \vec{a}_1 \mid \dots \mid \vec{b} \mid \dots \mid \vec{a}_n \,]$$

③ From determinant of matrix product:

$|A X_k| = |A| |X_k|$

Since $A X_k = A_k$ and $|X_k| = x_k$,

$|A_k| = |A| x_k$

Therefore,

$$x_k = \frac{|A_k|}{|A|}$$

Each component of the solution $\vec{x}$ can be computed by replacing one column of A at a time with $\vec{b}$:

$$x_1 = \frac{|A_1|}{|A|}$$

...

$$x_k = \frac{|A_k|}{|A|}$$

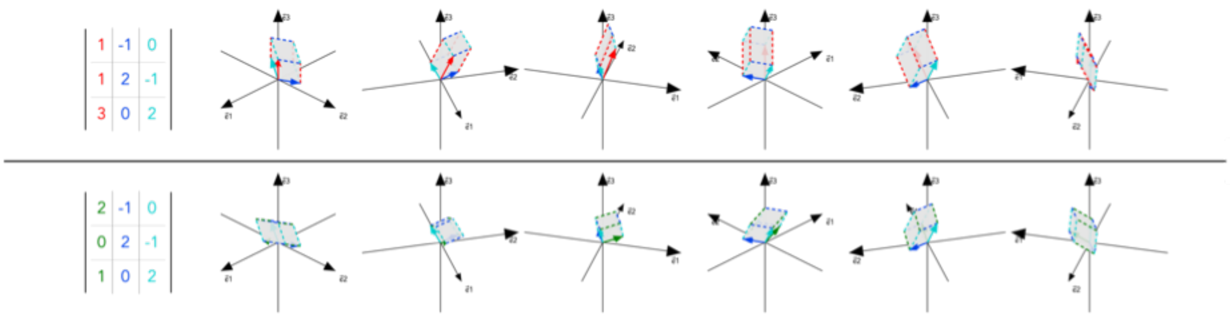
...

$$x_n = \frac{|A_n|}{|A|}$$

Cramer's rule: 3D illustration for the following system $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad |A| = 9$$

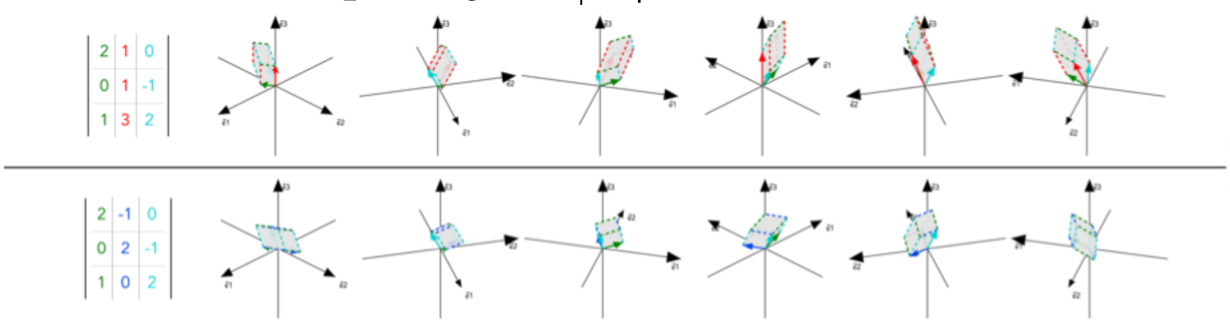
$x_1 = \dfrac{|A_1|}{|A|} =$



$= 1$

$\vec{b}$ and $\vec{a}_1$ have same-sign perpendicular components relative to the face formed by $\vec{a}_2$ and $\vec{a}_3$, so x_1 is positive

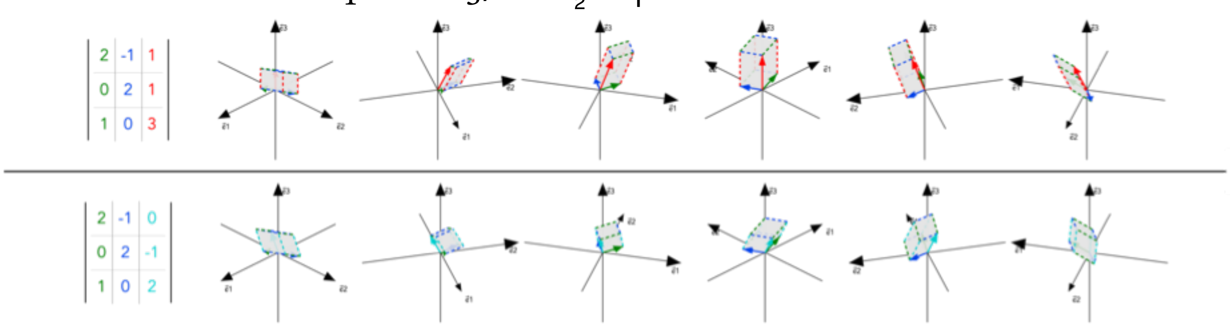
$x_2 = \dfrac{|A_2|}{|A|} =$



$= 1$

$\vec{b}$ and $\vec{a}_2$ have same-sign perpendicular components relative to the face formed by $\vec{a}_1$ and $\vec{a}_3$, so x_2 is positive

$x_3 = \dfrac{|A_3|}{|A|} =$



$= 1$

$\vec{b}$ and $\vec{a}_3$ have same-sign perpendicular components relative to the face formed by $\vec{a}_1$ and $\vec{a}_2$, so x_3 is positive

- Each numerator matrix A_k is obtained from A by replacing column k with $\vec{b}$

$$\bullet\ x_k = \frac{|A_k|}{|A|} \text{ is a signed volume ratio}$$

- the sign of x_k is determined by whether $\vec{b}$ and $\vec{a}_k$ have same-sign or opposite-sign perpendicular components relative to the plane spanned by the other columns