Change of basis: definition

Standard basis for $\mathbb{R}^n$

$$[\ \vec{e}_1 \mid \vec{e}_2 \mid \dots \mid \vec{e}_n\ ]$$

Suppose A is a square full-column-rank matrix:

$$A = [\ \vec{a}_1 \mid \vec{a}_2 \mid \dots \mid \vec{a}_n\ ]$$

Any $\vec{x}$ in $\mathbb{R}^n$ can be written as a linear combination of $\vec{a}_i$

$\leftrightarrow$

$\vec{x}$ can be written in basis A as $\vec{x}_A$

Change of coordinates from standard basis to basis A is a transformation that maps

- $\vec{a}_1 \rightarrow \vec{e}_1$
- $\vec{a}_2 \rightarrow \vec{e}_2$

...

- $\vec{a}_n \rightarrow \vec{e}_n$

If we denote this transformation matrix as B:

- $B\,\vec{a}_1 = \vec{e}_1$
- $B\,\vec{a}_2 = \vec{e}_2$

...

- $B\,\vec{a}_n = \vec{e}_n$

↓

$$B \left[\vec{a}_1 \mid \vec{a}_2 \mid \dots \mid \vec{a}_n \right] = \left[\vec{e}_1 \mid \vec{e}_2 \mid \dots \mid \vec{e}_n \right] = I_n$$

↓

$$B\,A = I_n$$

↓

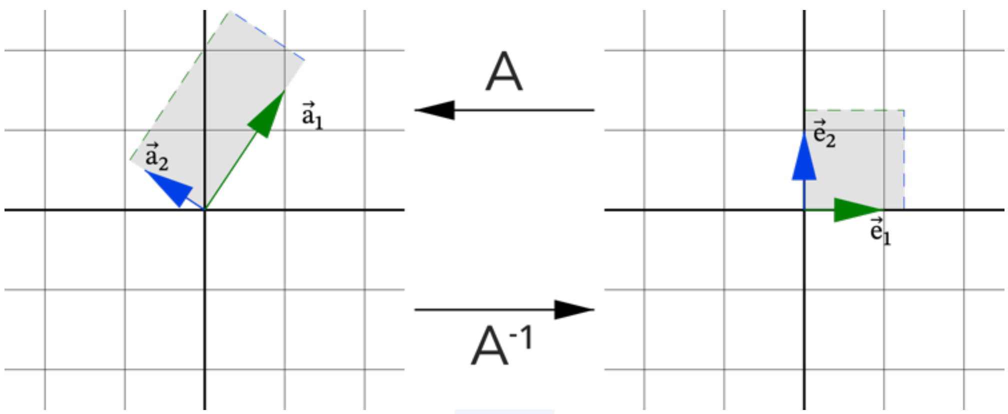
$$B = A^{-1}$$

↓

- $\vec{x}_A = A^{-1}\,\vec{x}$
- $\vec{x} = A\,\vec{x}_A$

Example: basis matrix A =

1	1.5
-0.75	0.5



Same vector $\vec{x}$ (gray dot) in two coordinate systems:

- standard basis $(\vec{e}_1, \vec{e}_2)$: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

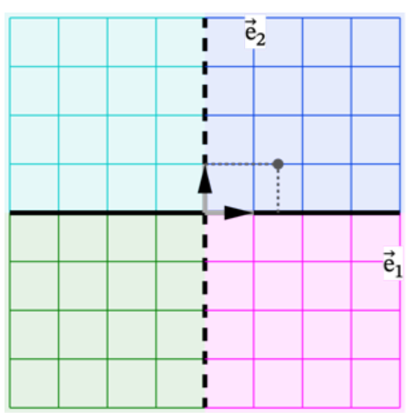
- basis $(\vec{a}_1, \vec{a}_2)$: $\vec{x}_A = \begin{bmatrix} (x_A)_1 \\ (x_A)_2 \end{bmatrix}$ with $A = [\, \vec{a}_1 \mid \vec{a}_2 \,]$

$\vec{x}$ expressed as a linear combination of $\vec{a}_1$ and $\vec{a}_2$:

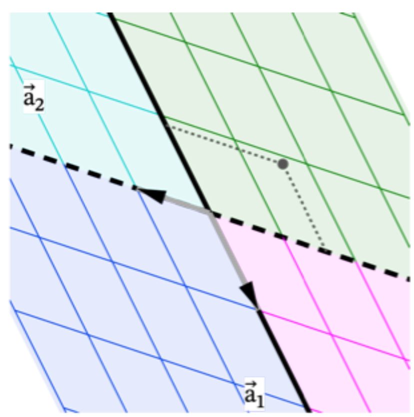
$$\vec{x} = \vec{a}_1 (x_A)_1 + \vec{a}_2 (x_A)_2$$

$$\downarrow$$

$$\vec{x} = [\, \vec{a}_1 \mid \vec{a}_2 \,] \begin{bmatrix} (x_A)_1 \\ (x_A)_2 \end{bmatrix} = A\, \vec{x}_A$$

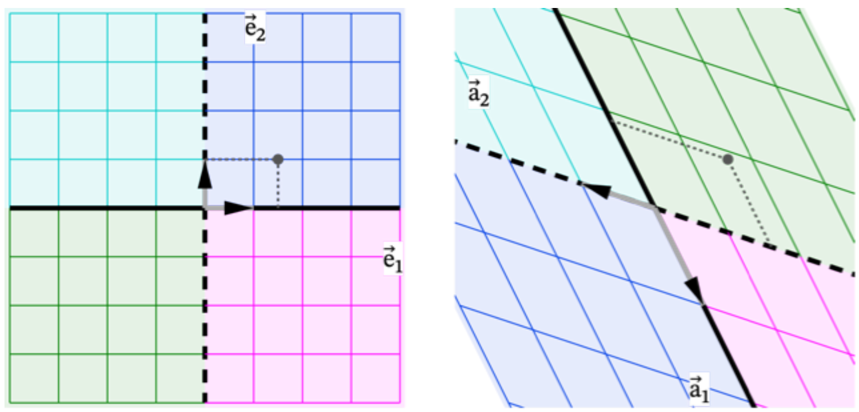


Standard basis $(\vec{e}_1, \vec{e}_2)$



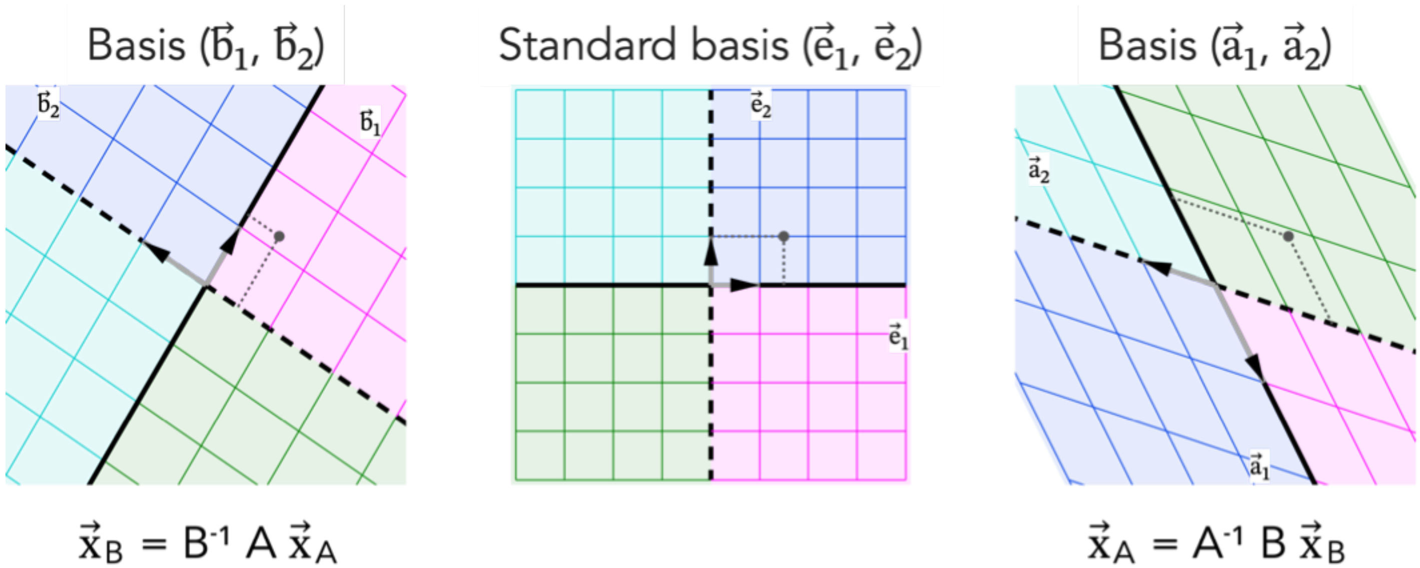
Basis $(\vec{a}_1, \vec{a}_2)$

$$\vec{x} = [\, \vec{a}_1 \mid \vec{a}_2 \,] \begin{bmatrix} (x_A)_1 \\ (x_A)_2 \end{bmatrix} = A\,\vec{x}_A$$

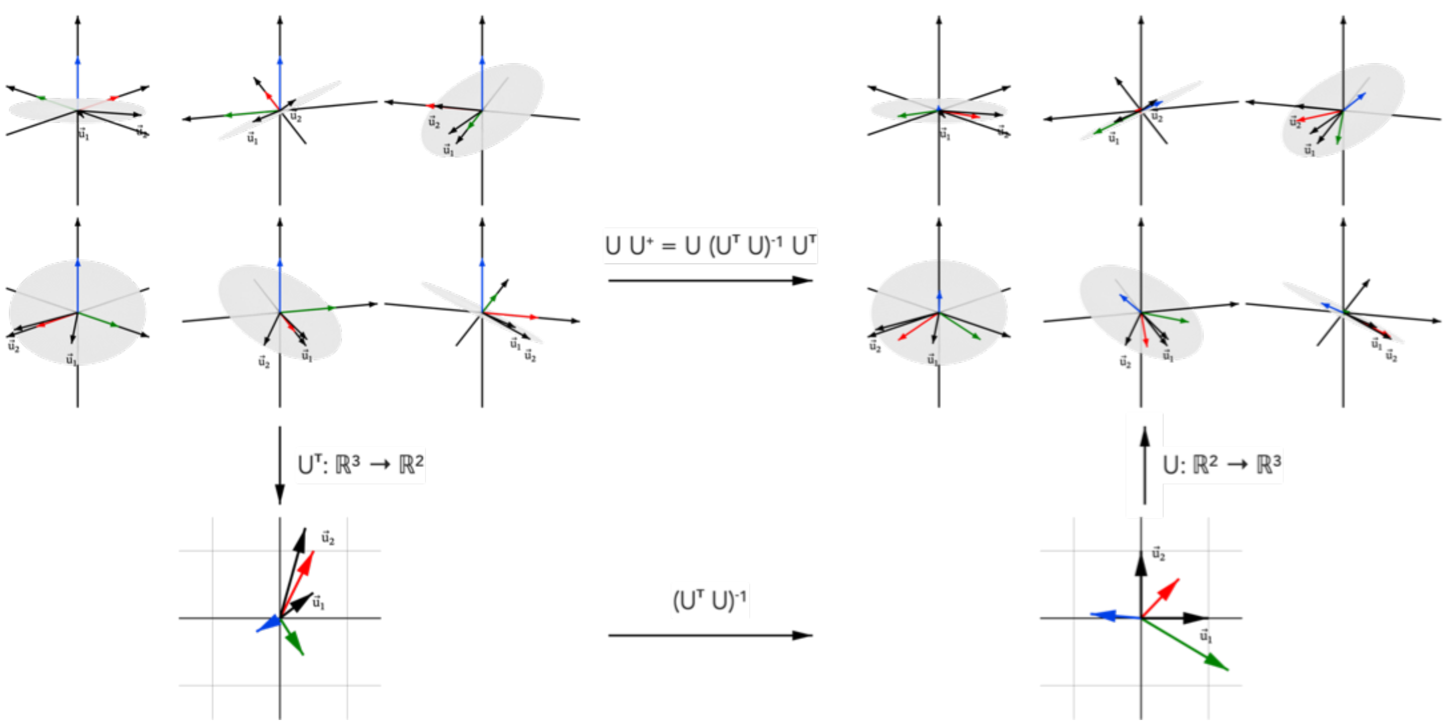


Standard basis ($\vec{e}_1$, $\vec{e}_2$) Basis ($\vec{a}_1$, $\vec{a}_2$)

$$B\,\vec{x}_B = [\, \vec{b}_1 \mid \vec{b}_2 \,] \begin{bmatrix} (x_B)_1 \\ (x_B)_2 \end{bmatrix} = \quad \vec{x}_E \quad = [\, \vec{a}_1 \mid \vec{a}_2 \,] \begin{bmatrix} (x_A)_1 \\ (x_A)_2 \end{bmatrix} = A\,\vec{x}_A$$



Change of basis with dimension lowering



Red, green and blue arrows: current images of standard basis vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$

Black arrows: current images of the spanning directions $\vec{u}_1$ and $\vec{u}_2$

U^T followed by $(U^T U)^{-1}$ gives the left inverse

$U^+ = (U^T U)^{-1} U^T$

U^+ maps $\vec{u}_1 \rightarrow \vec{e}_1$ and $\vec{u}_2 \rightarrow \vec{e}_2$,

so it changes vectors in $\mathbb{R}^3$ to u-basis coordinates in $\mathbb{R}^2$

Then U maps those u-basis coordinates back into $\mathbb{R}^3$

The round trip $U U^+$ is the projection matrix onto col(U):

$P = U U^+ = U (U^T U)^{-1} U^T$

Why change basis?

Change of basis is used to move matrices or vectors into coordinates where the computation becomes

- simpler due to either matrix structure or lower dimension
- more numerically stable

General principle for basis matrix B:

① move to a better basis with B^{-1}

② compute there

③ move back to the original basis with B

Common applications:

① Compute powers and exponentials

Find basis B where A is diagonal: $A = B\,D\,B^{-1}$

compute as

- $A^k = B\,D^k\,B^{-1}$
- $e^A = B\,e^D\,B^{-1}$

② Solve either consistent or inconsistent system

$A\,\vec{x} = \vec{b}$

If A is presented as Q R,

numerical error can be mitigated by solving

$R\,\vec{x} = Q^T\,\vec{b}$ in the orthonormal basis Q