$[\hat{k}]_\times$: cross product matrix for unit vector $\hat{k}$

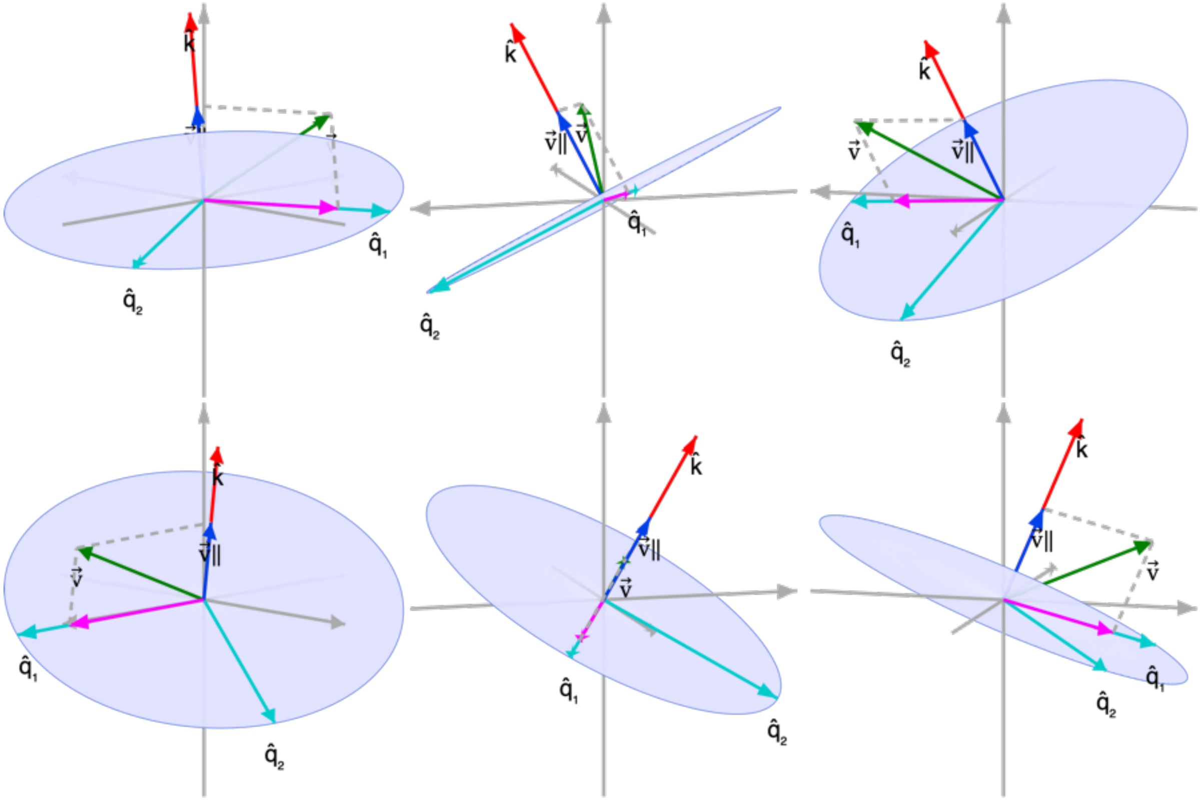
Matrix $[\hat{k}]_\times$ as a composition of the following transformations:
① removes the projection of $\vec{v}$ onto $\hat{k}$, computed as $\hat{k}\,\hat{k}^T\,\vec{v}$
↔
projects $\vec{v}$ onto plane orthogonal to $\hat{k}$
$(I - \hat{k}\,\hat{k}^T)\,\vec{v}$
equal to the projection matrix onto the plane spanned by orthonormal columns of Q:
$I - \hat{k}\,\hat{k}^T = Q\,Q^T$

Image below shows

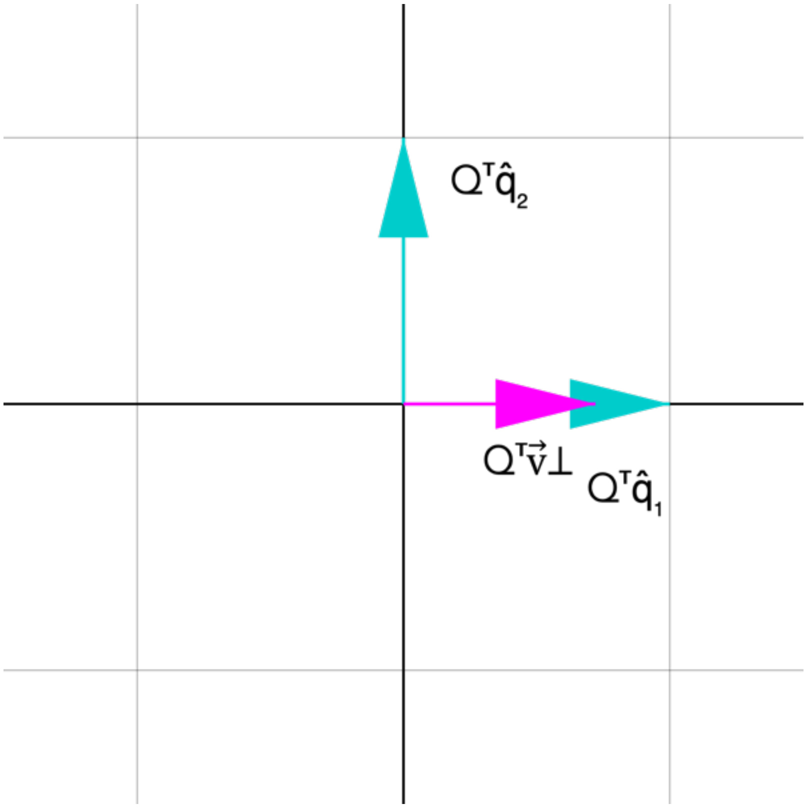
- $\hat{k}$ (red)
- $\vec{v}$ (green)
- $\vec{v}_\parallel = \hat{k}\,\hat{k}^T\,\vec{v}$ (blue)
- $\vec{v}_\perp = (I - \hat{k}\,\hat{k}^T)\,\vec{v}$ (magenta)
- $\hat{q}_1$ (cyan), can be computed as $\dfrac{\vec{v}_\perp}{|\vec{v}_\perp|}$
- $\hat{q}_2$ (cyan), computed from system of equations $\begin{bmatrix}\hat{q}_1 \\ \hat{k}\end{bmatrix}\vec{q} = \vec{0}$

$\vec{q}$ sign chosen so that $(\hat{q}_1, \vec{q}, \hat{k})$ is positively oriented

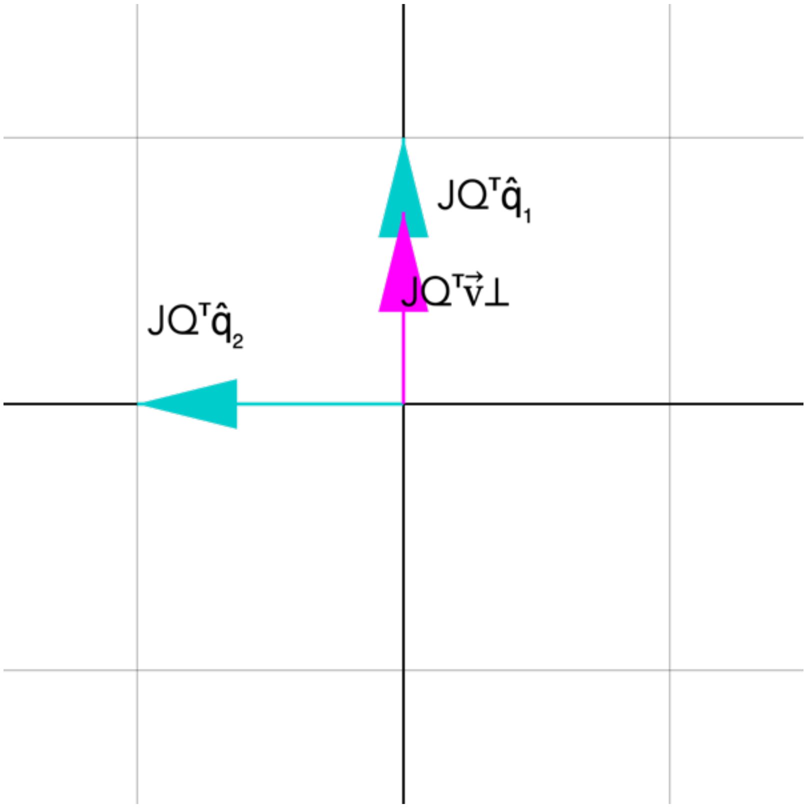
and normalized as $\hat{q}_2 = \dfrac{\vec{q}}{|\vec{q}|}$



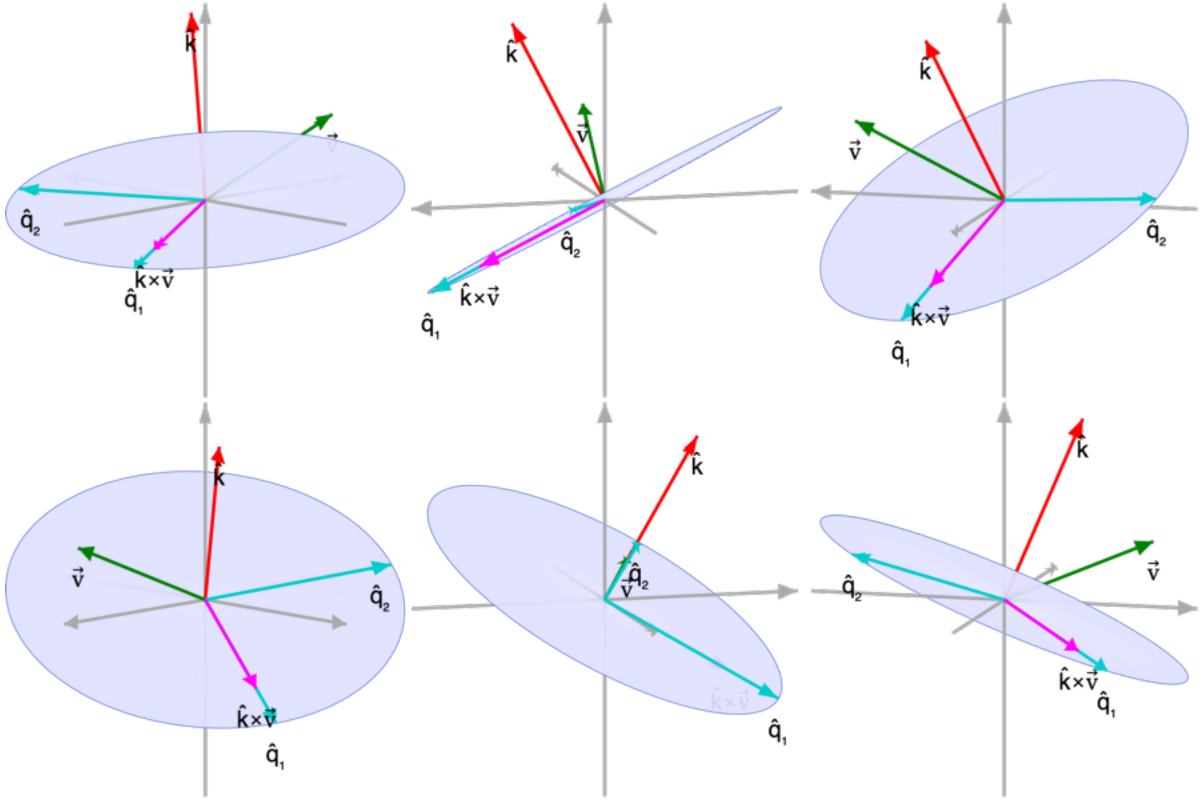
② expresses vectors in the orthogonal plane in ℝ² coordinates: $Q^T = \begin{bmatrix} \hat{q}_1^T \\ \hat{q}_2^T \end{bmatrix}$



③ applies the 90° rotation: $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



④ lifts back to $\mathbb{R}^3$: $Q = \left[\begin{array}{c|c} \hat{q}_1 & \hat{q}_2 \end{array}\right]$



$$(I - \hat{k}\,\hat{k}^T) = Q\,Q^T$$

$$\downarrow$$

$$[\,\hat{k}\,]\times = Q\,J\,Q^T\,(I - \hat{k}\,\hat{k}^T) =$$

$$Q\,J\,(Q^T\,Q)\,Q^T = Q\,J\,Q^T$$

$$Q\,J = \left[\begin{array}{c|c}\hat{q}_1 & \hat{q}_2\end{array}\right]\left[\begin{array}{c|c}0 & -1\\ \hline 1 & 0\end{array}\right] = \left[\begin{array}{c|c}\hat{q}_2 & -\hat{q}_1\end{array}\right]$$

$$[\,\hat{k}\,]\times = (Q\,J)\,Q^T = \left[\begin{array}{c|c}\hat{q}_2 & -\hat{q}_1\end{array}\right]\left[\begin{array}{c}\hat{q}_1^T\\ \hat{q}_2^T\end{array}\right] = \hat{q}_2\,\hat{q}_1^T - \hat{q}_1\,\hat{q}_2^T$$

The algebraic manipulations shown below confirm expected geometric actions of $[\,\hat{k}\,]\times$

- $[\,\hat{k}\,]\times\,\hat{k} = (\hat{q}_2\,\hat{q}_1^T - \hat{q}_1\,\hat{q}_2^T)\,\hat{k} =$

$\hat{q}_2 (\hat{q}_1^\top \hat{k}) - \hat{q}_1 (\hat{q}_2^\top \hat{k}) = \vec{0}$

↔

maps $\hat{k}$ to $\vec{0}$

- $[\, \hat{k}\,]\times\ \hat{q}_1 = (\hat{q}_2\, \hat{q}_1^\top - \hat{q}_1\, \hat{q}_2^\top)\, \hat{q}_1 =$
$\hat{q}_2 (\hat{q}_1^\top \hat{q}_1) - \hat{q}_1 (\hat{q}_2^\top \hat{q}_1) =$
$\hat{q}_2 - 0\, \hat{q}_1 = \hat{q}_2$

↔

rotates $\hat{q}_1 \rightarrow \hat{q}_2$

- $[\, \hat{k}\,]\times\ \hat{q}_2 = (\hat{q}_2\, \hat{q}_1^\top - \hat{q}_1\, \hat{q}_2^\top)\, \hat{q}_2 =$
$\hat{q}_2 (\hat{q}_1^\top \hat{q}_2) - \hat{q}_1 (\hat{q}_2^\top \hat{q}_2) =$
$\hat{q}_2\, 0 - \hat{q}_1 = - \hat{q}_1$

↔

rotates $\hat{q}_2 \rightarrow -\hat{q}_1$

The above identities uniquely describe
how $[\, \hat{k}\,]\times$ transforms the 3 spanning vectors

Note the particular form of the matrix, called skew-symmetric

$$\hat{q}_2\, \hat{q}_1^\top - \hat{q}_1\, \hat{q}_2^\top = \left[\begin{array}{c|c|c} 0 & q_{12}\times q_{21} - q_{11}\times q_{22} & q_{12}\times q_{31} - q_{11}\times q_{32} \\ \hline q_{22}\times q_{11} - q_{21}\times q_{12} & 0 & q_{22}\times q_{31} - q_{21}\times q_{32} \\ \hline q_{32}\times q_{11} - q_{31}\times q_{12} & q_{32}\times q_{21} - q_{31}\times q_{22} & 0 \end{array}\right]$$

On the next page we will derive
the unique matrix that satisfies all the geometric properties
independent of choice of $\hat{q}_1$ & $\hat{q}_2$

$$[\,\hat{k}\,]\times = Q\,J\,Q^{T} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

Computing entries of $[\,\hat{k}\,]\times$

As was shown above, $[\,\hat{k}\,]\times$ has a skew-symmetric form:

$$\hat{q}_2\,\hat{q}_1^{T} - \hat{q}_1\,\hat{q}_2^{T} = \begin{bmatrix} 0 & q_{12}\times q_{21} - q_{11}\times q_{22} & q_{12}\times q_{31} - q_{11}\times q_{32} \\ q_{22}\times q_{11} - q_{21}\times q_{12} & 0 & q_{22}\times q_{31} - q_{21}\times q_{32} \\ q_{32}\times q_{11} - q_{31}\times q_{12} & q_{32}\times q_{21} - q_{31}\times q_{22} & 0 \end{bmatrix}$$

The above matrix can be re-written as $\left([\,\hat{k}\,]\times\right) = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$

Recall geometric actions of $[\,\hat{k}\,]\times$:

- maps $\hat{k}$ to $\vec{0}$
- rotates $\hat{q}_1 \rightarrow \hat{q}_2$
- rotates $\hat{q}_2 \rightarrow -\hat{q}_1$

↓

$\text{col}\left([\,\hat{k}\,]\times\right)$ is a plane in $\mathbb{R}^3{}_1$

↓

3×3 matrix $[\,\hat{k}\,]\times$ has rank = 2

↓

$\text{null}\left([\,\hat{k}\,]\times\right)$ is a line in $\mathbb{R}^3$

↓

any vector in $\text{null}\left([\,\hat{k}\,]\times\right)$ is a scalar multiple of $\hat{k}$

From algebraic properties of skew-symmetric $\left([\,\hat{k}\,]\times\right)$, we can infer that

$$\text{null}\left([\,\hat{k}\,]\times\right) \text{ is spanned by a vector } \vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

↓

same 1-dimensional space is spanned by $\hat{k}$ & $\vec{a}$

↓

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (d\,\hat{k}) \text{ where d is a scalar}$$

Next, we need to find the unique d that satisfies

- $\left([\,\hat{k}\,]\times\right)\hat{q}_1 = \hat{q}_2$
- $\left([\,\hat{k}\,]\times\right)\hat{q}_2 = -\hat{q}_1$

We will consider a simple case of $\mathbb{R}^3$ where

- $\vec{e}_3 = \hat{k}$
- $\vec{e}_1 = \hat{q}_1$
- $\vec{e}_2 = \hat{q}_2$

In this case,

- $d \left([\,\hat{k}\,]\times \right) \vec{e}_1 = \vec{e}_2$
- $d \left([\,\hat{k}\,]\times \right) \vec{e}_1 = d \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$d = 1$ is the only scalar that satisfies the above

↓

$$[\,\hat{k}\,]\times = Q\,J\,Q^{T} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

Summary:

① geometry provided us the axis is the null space
algebra indicated that the null space is generated by vector $\vec{a} = [a\ b\ c]^{T}$
both null spaces are 1-dimensional

↓

they are the same object:

$$\text{null}\left([\,\hat{k}\,]\times \right) = \text{span}(\hat{k}) = \text{span}(\vec{a})$$

↓

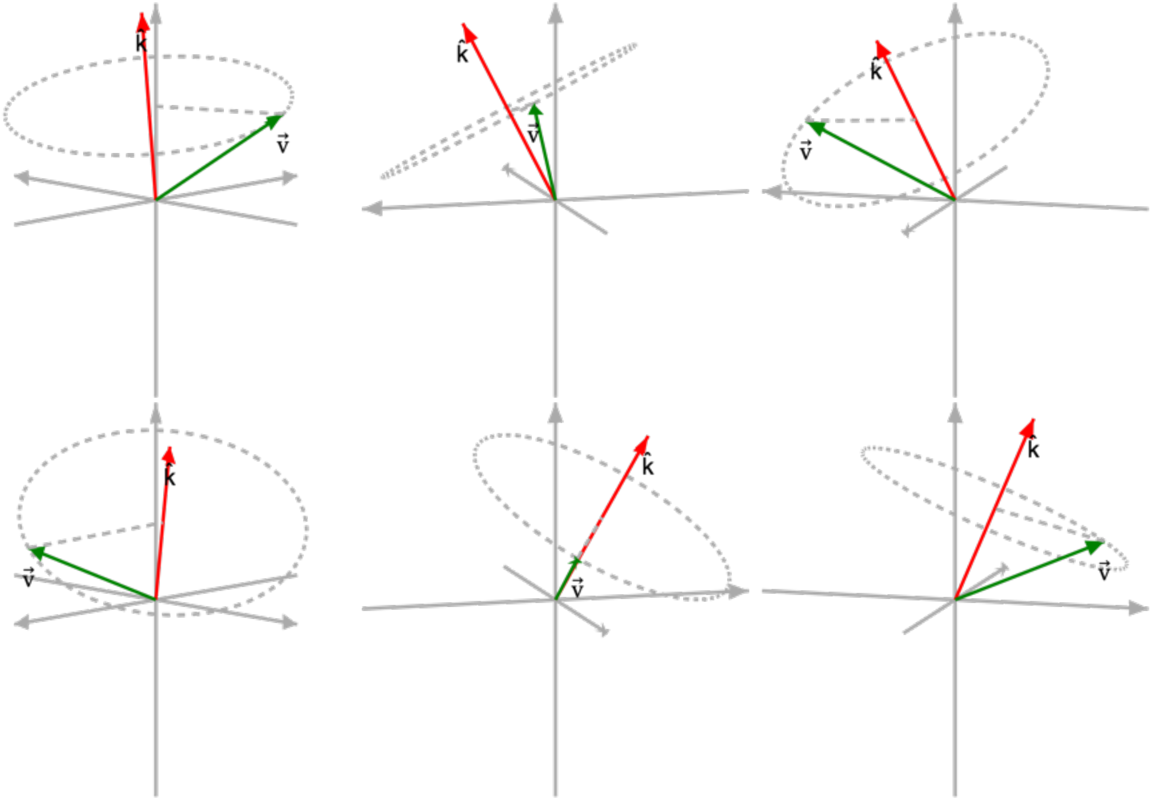
$\vec{a} = d\,\hat{k}$ for some scalar d

② we used another geometric property, considered the simplest case and obtained $d = 1$

③ even though the derivation employed non-unique matrix $Q = [\,\hat{q}_1 \mid \hat{q}_2]$, $[\,\hat{k}\,]_\times = Q\,J\,Q^T$ is independent of Q

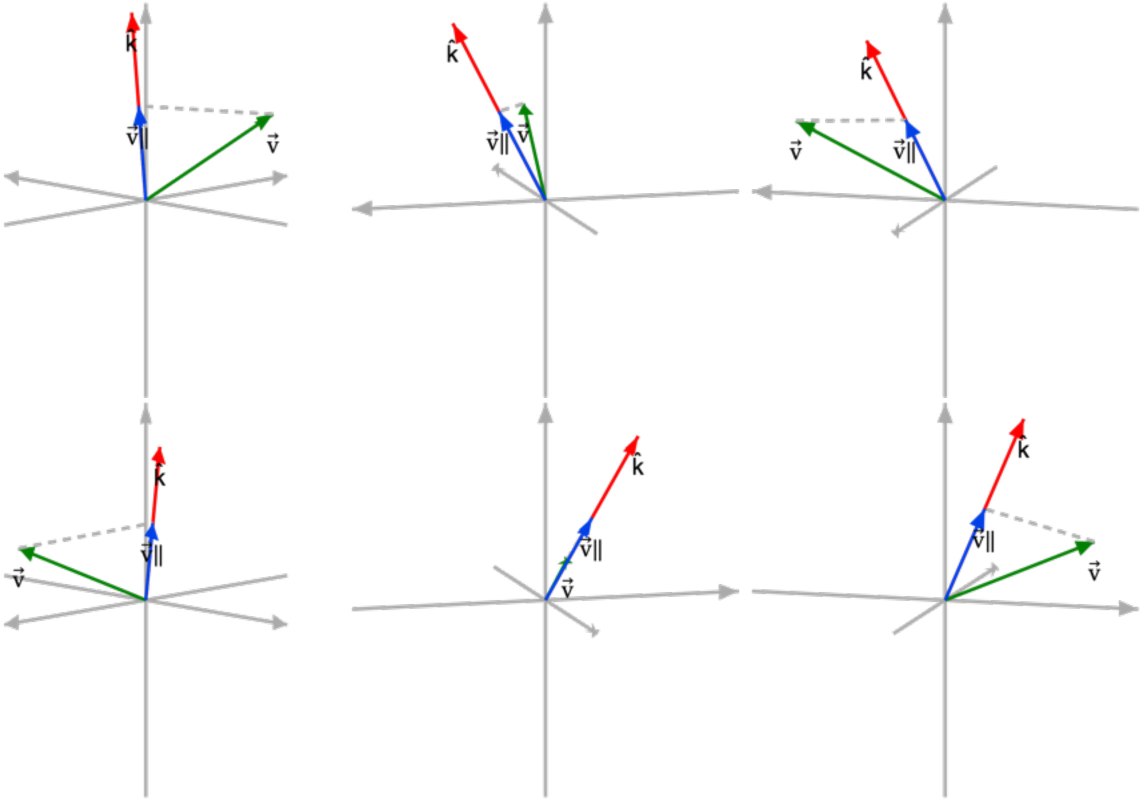
Rotation around a vector (Rodrigues algorithm)

Problem stated:
rotate vector $\vec{v}$ (green) around unit vector $\hat{k}$ (red) by angle θ
where $\vec{v}$ & $\hat{k}$ are non-colinear



① Calculate projection of $\vec{v}$ onto $\hat{k}$ as

$\vec{v}|| = \hat{k}\,\hat{k}^T\,\vec{v}$ (blue)



② Define a unit vector $\hat{q}_1$ (cyan) so that $\hat{q}_1 \cdot \hat{k} = 0$

while there is not a unique choice for $\hat{q}_1$,
one particular instance can be computed by normalizing $\vec{v}\perp = \vec{v} - \vec{v}||$

$$\hat{q}_1 = \frac{\vec{v}\perp}{|\,\vec{v}\perp\,|}$$

③ Define another unit vector $\hat{q}_2$ (cyan) such that

- $\hat{q}_2 \cdot \hat{q}_1 = 0$
- $\hat{q}_2 \cdot \hat{k} = 0$

- equivalently $$\begin{bmatrix} \hat{k}^T \\ \hat{q}_1^T \end{bmatrix} \vec{x} = \vec{0}$$

$\hat{k}$ and $\hat{q}_1$ are orthogonal (linealy independent) vectors in $\mathbb{R}^3$

↓

solution space is one-dimensional

- choose any non-zero $\vec{x}$ so that $(\hat{q}_1, \vec{x}, \hat{k})$ is positively oriented
- normalize as $\hat{q}_2 = \dfrac{\vec{x}}{|\vec{x}|}$

↓

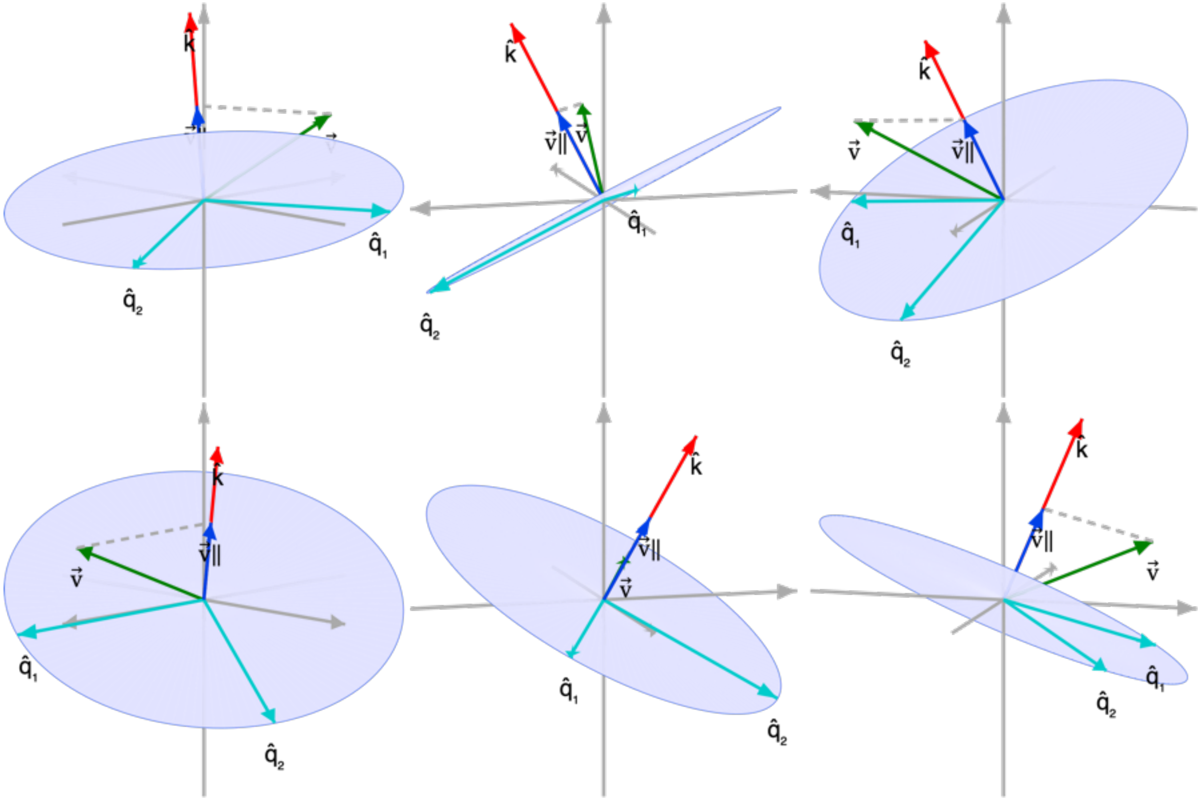
once $\hat{q}_1$ is chosen, $\hat{q}_2$ is uniquely determined up to the sign

(alternatively, $\hat{q}_2$ can be computed as cross product: $\hat{q}_2 = \hat{k} \times \hat{q}_1$)

④ $\hat{q}_1$ & $\hat{q}_2$ now span the plane orthogonal to $\hat{k}$
and we have $Q = [\ \hat{q}_1 \mid \hat{q}_2\ ]$
Q is orthonormal

↓

$(Q\,Q^T)$ is the matrix of projection onto Q



⑤ $\vec{v}\perp$ (magenta) is the projection of $\vec{v}$ onto the plane

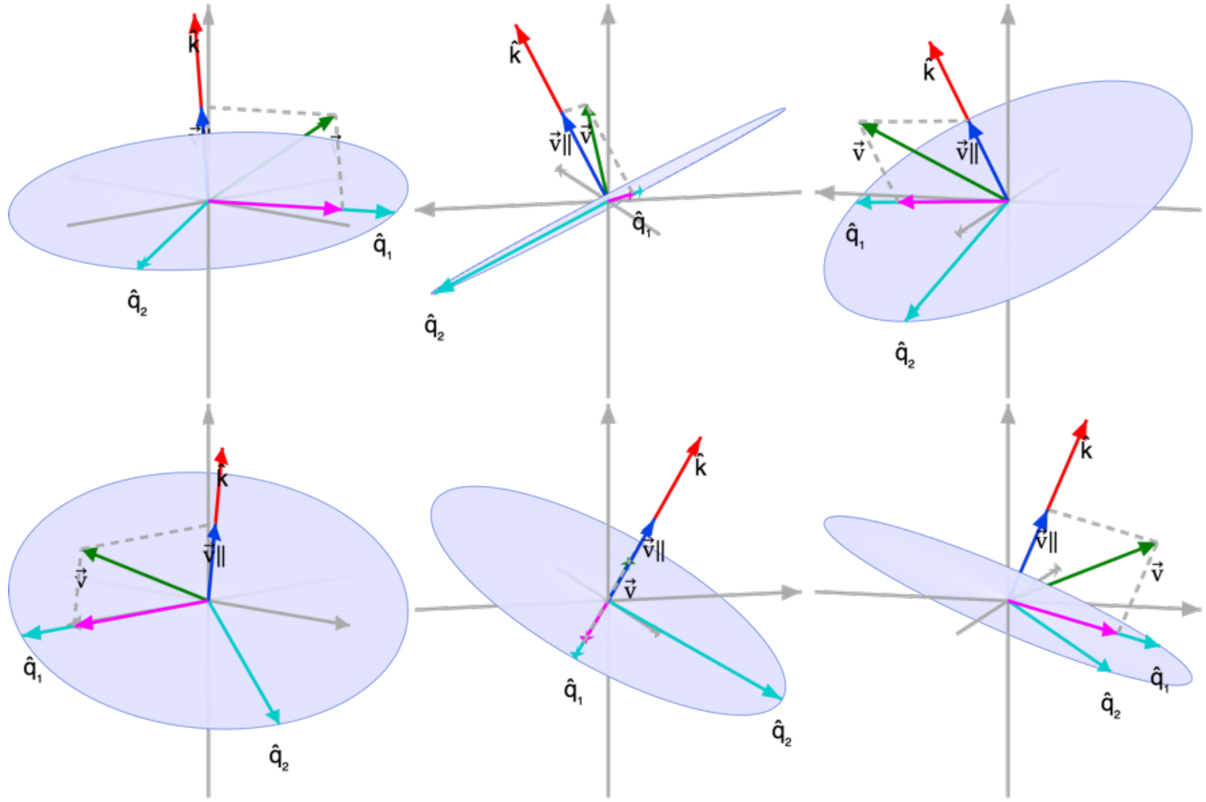
- $\vec{v}\perp = \vec{v} - \vec{v}\|$

$$\downarrow$$

$$\vec{v}\perp = (I - \hat{k}\,\hat{k}^T)\,\vec{v}$$

Also, $\vec{v}\perp = Q\,Q^T\,\vec{v}$

As $\vec{v}$ rotates around $\hat{k}$,
- $\vec{v}\|$ stays the same
- $\vec{v}\perp$ rotates inside the plane to become $\vec{w}\perp$



⑥ Represent the plane via 2D coordinates as multiplication by Q^T ($\mathbb{R}^3 \rightarrow \mathbb{R}^2$):

- $Q^T\,\hat{q}_1 = \begin{bmatrix} \hat{q}_1^T \\ \hat{q}_2^T \end{bmatrix} \hat{q}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hat{e}_1(2D)$

- $Q^{T}\hat{q}_2 = \begin{bmatrix} \hat{q}_1^{T} \\ \hat{q}_2^{T} \end{bmatrix} \hat{q}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hat{e}_2(2D)$

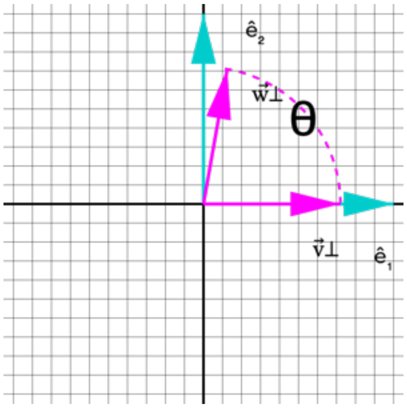
- $\vec{v}\perp(3D) = s\,\hat{q}_1$ (s is a scalar)

$\downarrow$

- $\vec{v}\perp(2D) = s\,\hat{e}_1$

⑦ $\vec{v}\perp(2D)$ rotated inside the plane by θ becomes $\vec{w}\perp(2D)$

$$\vec{w}\perp(2D) = R(\theta)\ \left(\vec{v}\perp(2D)\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \left(\vec{v}\perp(2D)\right)$$



⑧ Represent $\vec{w}\perp$ in 3D coordinates

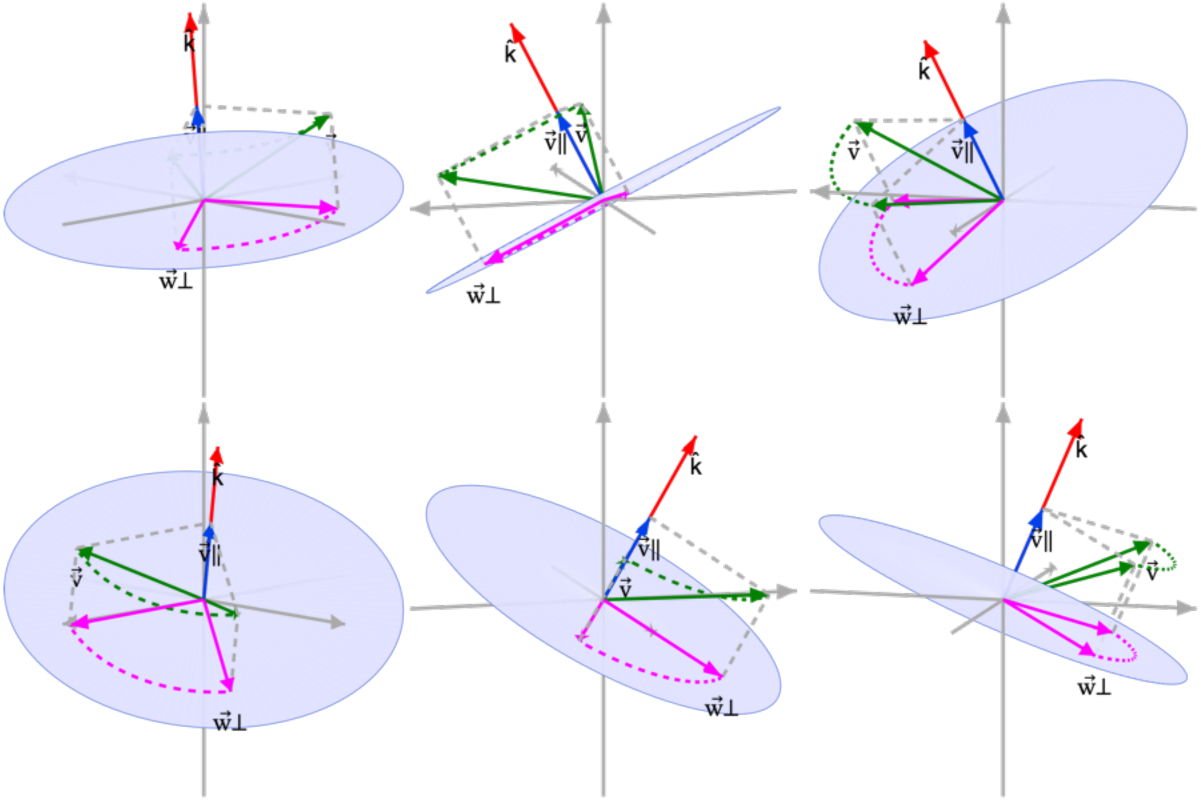
$Q\ (\mathbb{R}^2 \rightarrow \mathbb{R}^3)$

(from plane to 3D coordinates)

- $Q\,\hat{e}_1 = \hat{q}_1$
- $Q\,\hat{e}_2 = \hat{q}_2$

- $Q\ (\vec{w}\perp \in \mathbb{R}^2) \rightarrow (\vec{w}\perp \in \mathbb{R}^3)$

⑨ Reconstruct $\vec{w} = \vec{w}\perp + \vec{v}\|$ (green)



⑩ Combining these steps gives the following

$$\vec{w} = \hat{k}\,\hat{k}^T\,\vec{v} + Q\,R(\theta)\,Q^T\,(I - \hat{k}\,\hat{k}^T)\,\vec{v}$$

- $\hat{k}\,\hat{k}^T\,\vec{v}$ is the unchanged projection onto $\hat{k}$
- $(I - \hat{k}\,\hat{k}^T)\,\vec{v}$ is the component of $\vec{v}$ inside the plane or $\vec{v}\perp$
- $Q\,R(\theta)\,Q^T\,(I - \hat{k}\,\hat{k}^T) =$
  $Q\,R(\theta)\,(Q^T Q)\,Q^T = Q\,R(\theta)\,Q^T$
  is the sequence of transformations applied to $\vec{v}\perp$

While this geometric construction is sufficient to compute the rotated vector $\hat{w}$, the standard closed-form expression

$$\hat{w} = A\,\vec{v} = \left[\cos\theta\, I + (1 - \cos\theta)\,\hat{k}\,\hat{k}^T + \sin\theta\,[\hat{k}]\times\right]\vec{v}$$

will be derived on the next page
as a compact representation of the same operations,
useful for algebraic manipulation and computational efficiency

Matrix A derived from above algorithm

On the previous page, we showed that

$\vec{w} = \vec{v}|| + Q\ R(\theta)\ Q^{T}\ \vec{v}\perp =$

$\hat{k}\ \hat{k}^{T}\ \vec{v} + Q\ R(\theta)\ Q^{T}\ (I - \hat{k}\ \hat{k}^{T})\ \vec{v}$

- axis of rotation: unit vector $\hat{k} = [\ k_1, k_2, k_3\ ]^{T}$
- $\vec{v}$ is a vector we want to rotate around $\hat{k}$
- $\hat{k}\ \hat{k}^{T}\ \vec{v}$ is the projection of $\vec{v}$ onto span($\hat{k}$)
- $Q = [\ \hat{q}_1 \mid \hat{q}_2\ ]$ is a 3×2 matrix with orthonormal columns such that $\hat{q}_1 \cdot \hat{k} = 0$, $\hat{q}_2 \cdot \hat{k} = 0$
- $Q\ Q^{T}$ is the matrix of projection onto col(Q)
- $(I - \hat{k}\ \hat{k}^{T})\ \vec{v}$ is also projection of $\vec{v}$ onto col(Q)

$$\downarrow$$

$$Q\ R(\theta)\ Q^{T}\ (I - \hat{k}\ \hat{k}^{T}) =$$

$$Q\ R(\theta)\ (Q^{T}\ Q)\ Q^{T} =$$

$$Q\ R(\theta)\ Q^{T}$$

$$\downarrow$$

$$\vec{w} = \hat{k}\ \hat{k}^{T}\ \vec{v} + Q\ R(\theta)\ Q^{T}\ \vec{v}$$

- $R(\theta) = \left[\begin{array}{c|c} \cos\theta & -\sin\theta \\ \hline \sin\theta & \cos\theta \end{array}\right]$

$$Q\ R(\theta)\ Q^{T} = \left[\begin{array}{c|c} \hat{q}_1 & \hat{q}_2 \end{array}\right] \left[\begin{array}{c|c} \cos\theta & -\sin\theta \\ \hline \sin\theta & \cos\theta \end{array}\right] \left[\begin{array}{c} \hat{q}_1 \\ \hat{q}_2 \end{array}\right] =$$

$$\left[\begin{array}{c|c} \hat{q}_1 & \hat{q}_2 \end{array}\right] \begin{bmatrix} \cos\theta\,\hat{q}_1 - \sin\theta\,\hat{q}_2 \\ \sin\theta\,\hat{q}_1 + \cos\theta\,\hat{q}_2 \end{bmatrix} =$$

$$\hat{q}_1\,(\cos\theta\,\hat{q}_1 - \sin\theta\,\hat{q}_2) + \hat{q}_2\,(\sin\theta\,\hat{q}_1 + \cos\theta\,\hat{q}_2) =$$
$$\cos\theta\,(\hat{q}_1\,\hat{q}_1^{\top} + \hat{q}_2\,\hat{q}_2^{\top}) + \sin\theta\,(\hat{q}_2\,\hat{q}_1^{\top} - \hat{q}_1\,\hat{q}_2^{\top})$$

Sum of outer products

$$\hat{q}_1\,\hat{q}_1^{\top} + \hat{q}_2\,\hat{q}_2^{\top} = Q\,Q^{\top}$$
$$\downarrow$$
$$Q\,R(\theta)\,Q^{\top} = \cos\theta\,Q\,Q^{\top} + \sin\theta\,(\hat{q}_2\,\hat{q}_1^{\top} - \hat{q}_1\,\hat{q}_2^{\top})$$
$$\downarrow$$
$$\vec{w} = \hat{k}\,\hat{k}^{\top}\,\vec{v} + \cos\theta\,(I - \hat{k}\,\hat{k}^{\top})\,\vec{v} + \sin\theta\,(\hat{q}_2\,\hat{q}_1^{\top} - \hat{q}_1\,\hat{q}_2^{\top})\,\vec{v}$$

So the full 3×3 rotation matrix is

$$A = \hat{k}\,\hat{k}^{\top} + \cos\theta\,(I - \hat{k}\,\hat{k}^{\top}) + \sin\theta\,(\hat{q}_2\,\hat{q}_1^{\top} - \hat{q}_1\,\hat{q}_2^{\top}) =$$
$$\cos\theta\,I + (1 - \cos\theta)\,\hat{k}\,\hat{k}^{\top} + \sin\theta\,(\hat{q}_2\,\hat{q}_1^{\top} - \hat{q}_1\,\hat{q}_2^{\top})$$

In the previous page, we showed that

$\hat{q}_2\,\hat{q}_1^{\top} - \hat{q}_1\,\hat{q}_2^{\top}$ is the cross product matrix of $\hat{k}$

$$[\,\hat{k}\,]_{\times} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

and

$$A = \cos\theta\, I + (1 - \cos\theta)\, \hat{k}\, \hat{k}^T + \sin\theta \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

On the previous page, we demonstrated an algorithm that
uses normalized $\vec{v}\perp$ to compute one instance of $\hat{q}$
However, the derived matrix A is independent of choice of $\vec{v}$

Our tutorial used the following sequence of derivations:
- First, the cross product matrix $[\, \hat{k}\,]\times$ was identified
from the action of the particular 90° rotation in $\hat{k}\perp$
and the special structure of J = Rot(90°) made that identification possible
- Next, the rotation algorithm around $\hat{k}$ was built geometrically from projection onto $\hat{k}$,
rotation in $\hat{k}\perp$ and lifting back through the orthonormal basis $Q = [\, \hat{q}_1 \mid \hat{q}_2\,]$
- Finally, expanding that algorithm produced Rodrigues' formula,
with $[\, \hat{k}\,]\times$ appearing as the skew-symmetric term in the final matrix expression

---

Eigenvectors of A: matrix of rotation around $\hat{k}$

① Geometrically, $\hat{k}$ is not affected by A

We confirm with algebra:
$A\, \hat{k} =$

$$\left(\cos\theta\, I + (1 - \cos\theta)\, \hat{k}\, \hat{k}^T + \sin\theta\, (\hat{q}_2\, \hat{q}_1^T - \hat{q}_1\, \hat{q}_2^T)\right)\, \hat{k} =$$

$$\cos\theta\ \hat{k} + (1 - \cos\theta)\ \hat{k}\ (\hat{k}^T \hat{k}) + \sin\theta\ (\hat{q}_2\ \hat{q}_1^T - \hat{q}_1\ \hat{q}_2^T)\ \hat{k} =$$

$$\cos\theta\ \hat{k} + (1 - \cos\theta)\ \hat{k} + \sin\theta\ \left(\hat{q}_2\ (\hat{q}_1^T\ \hat{k}) - \hat{q}_1\ (\hat{q}_2^T\ \hat{k})\right) = \hat{k}$$

$$A\ \hat{k} = \hat{k}$$

$$\downarrow$$

$\hat{k}$ is an eigenvector of A with $\lambda = 1$

② The remaining action of A is a rotation inside the plane orthogonal to $\hat{k}$

Eigenvalue-eigenvector pairs of 2D rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

λ2D	$\vec{x}$
$e^{i\theta}$	$[\,1, -i\,]^T$
$e^{-i\theta}$	$[\,1, i\,]^T$

Previously, we showed that

$$A = \hat{k}\ \hat{k}^T + Q\ R\ Q^T$$

columns of Q are orthonormal and orthogonal to $\hat{k}$

$$\downarrow$$

$$\hat{k}^T\ Q = \vec{0} \text{ and } Q^T\ Q = I$$

Next, we show that

if $\vec{x}$ is an eigenvector of R, then $Q\ \vec{x}$ is an eigenvector of A:

$$A\ (Q\ \vec{x}) = \hat{k}\ \hat{k}^T\ Q\ \vec{x} + Q\ R\ Q^T\ Q\ \vec{x} =$$

$$\hat{k}\ (\hat{k}^T\ Q)\ \vec{x} + Q\ R\ (Q^T\ Q)\ \vec{x} =$$

$$\vec{0} + Q\,R\,\vec{x}$$

If $R\,\vec{x} = \lambda\,\vec{x}$

$$A\,(Q\,\vec{x}) = Q\,(R\,\vec{x}) = Q\,(\lambda\,\vec{x}) = \lambda\,(Q\,\vec{x})$$

$$\downarrow$$

- $Q\,\vec{x}$ is an eigenvector of A
- $\lambda(3D) = \lambda(2D)$

- $[\,\hat{q}_1 \,|\, \hat{q}_2\,] \begin{bmatrix} 1 \\ -i \end{bmatrix} = \hat{q}_1 - i\,\hat{q}_2$

- $[\,\hat{q}_1 \,|\, \hat{q}_2\,] \begin{bmatrix} 1 \\ i \end{bmatrix} = \hat{q}_1 + i\,\hat{q}_2$

Eigenvalue-eigenvector pairs of A

λ3D	$\vec{y} = Q\,\vec{x}$
1	$\hat{k}$
$e^{i\theta}$	$\hat{q}_1 - i\,\hat{q}_2$ (infinitely many depending on choice of $\hat{q}_1$ & $\hat{q}_2$)
$e^{-i\theta}$	$\hat{q}_1 + i\,\hat{q}_2$ (infinitely many depending on choice of $\hat{q}_1$ & $\hat{q}_2$)

$$\vec{v}_{\parallel} = \hat{k}\,\hat{k}^{T}\,\vec{v}$$

$$\vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel}$$

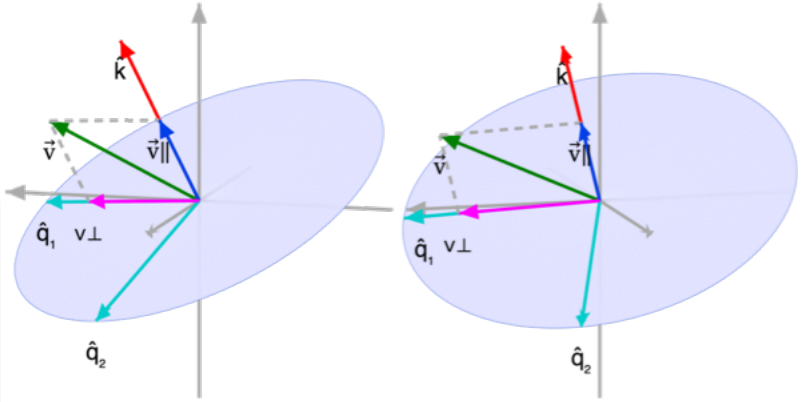
$$\hat{q}_1 = \frac{\vec{v}_{\perp}}{|\,\vec{v}_{\perp}\,|}$$

$\hat{q}_2 = \hat{k} \times \hat{q}_1$ (cross product of unit vectors)

$$Q = [\ \hat{q}_1 \mid \hat{q}_2\ ]$$

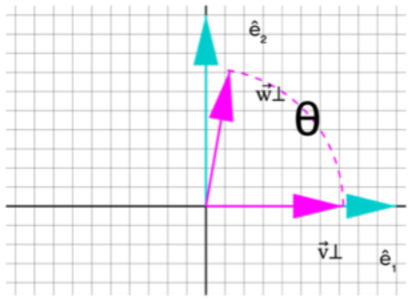
$$\vec{w} = \vec{v}\| + \cos\theta\, \vec{v}\perp + \sin\theta\, (\hat{k} \times \vec{v}\perp)$$

$$\vec{w} = \vec{v}\| + Q\, R(\theta)\, Q^T \vec{v}$$



Q^T ($\mathbb{R}^3 \rightarrow \mathbb{R}^2$)
(plane coordinates)
$Q^T (\vec{v}\perp \in \mathbb{R}^3) \rightarrow (\vec{v}\perp \in \mathbb{R}^2)$
$Q^T \hat{q}_1 = \hat{e}_1$
$Q^T \hat{q}_2 = \hat{e}_2$

$Q^T = [\, \hat{q}_1^T \mid \hat{q}_2^T \,]$
$\hat{q}_1 \cdot \hat{q}_2 = 0$
↓
Q^T preserves
length & angles



$$(\vec{w}\perp \in \mathbb{R}^2) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} (\vec{v}\perp \in \mathbb{R}^2)$$

Q ($\mathbb{R}^2 \rightarrow \mathbb{R}^3$)
(from plane coordinates)
$Q (\vec{w}\perp \in \mathbb{R}^2) \rightarrow (\vec{w}\perp \in \mathbb{R}^3)$
$Q\, \hat{e}_1 = \hat{q}_1$
$Q\, \hat{e}_2 = \hat{q}_2$

$Q = [\, \hat{q}_1 \mid \hat{q}_2 \,]$
$\hat{q}_1 \cdot \hat{q}_2 = 0$
↓
Q preserves
length & angles